Theory of Computation: Assignment 5 Solutions

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1. The "proof" claims that when you pump up, the 0s and 1s won't be equal and the new string will not be in the language. However, 0^*1^* does not require the 0s and 1s to be equal - we just need to 0s before the 1s. Pumping y (which contains all 0s) does not change this, thus the new string is still in 0^*1^*, and the pumping lemma is not contradicted.

2. (a) $L = \{0^n 1^n 2^n | n \geq 0\}$
   AFSOC L is regular with pumping length p. Take $w = 0^p 1^p 2^p = \underbrace{0 \ldots 0}_{p} \underbrace{1 \ldots 1}_{p} \underbrace{2 \ldots 2}_{p}$. Clearly $w \in L$ and $|w| \geq p$, so w should be pumpable. Split up $w = xyz$ where $|xy| \leq p$ and $|y| > 0$. Since $|xy| \leq p$, y contains only 0s. When we pump up y, there will be more 0s than 1s and 2s. The new string will not be in the language, which is a contradiction of the pumping lemma. Thus, L is not regular.

   (b) $L = \{www | w \in \{a, b\}^*\}$.
   AFSOC L is regular with pumping length p. Take $w = a^p b a^p b a^p b = \underbrace{a \ldots a}_{p} b \underbrace{a \ldots a}_{p} b \underbrace{a \ldots a}_{p} b$. Clearly $w \in L$ and $|w| \geq p$, so w should be pumpable. Split up $w = xyz$ where $|xy| \leq p$ and $|y| > 0$. Since $|xy| \leq p$, y contains only <u>leading</u> a's. When we pump up y, there will be more leading a's than middle or trailing a's. The new string will not be in the language, which is a contradiction of the pumping lemma. Thus, L is not regular.

   Note: It might be tempting to pick $w = a^p a^p a^p$, and argue that when we split it up and pump, only the leading a's will be pumped, and the string will not consist of three consecutive identical substrings. However, because this string consists of all a's, there are no "leading" or "trailing" a's; we can simply re-arrange the a's to try to balance out the three parts of the string..
   To make this more concrete: let $w = a^p a^p a^p$. Split up $w = xyz$ such that $|y| > 0$ and $|xy| \leq p$. If we pick $y = aaa$, then when we pump k times we will have the string

$$a^{p+3(k-1)} a^p a^p$$

   At first it looks like the a's in the first part of the string will not match the a's in the last two parts. However, note that:

$$a^{p+3(k-1)} a^p a^p = a^{p+(k-1)} a^{p+(k-1)} a^{p+(k-1)}$$

   This, the string $a^p a^p a^p$ is actually pumpable. By contrast, by picking $w = a^p b a^p b a^p b$, the b's act as a "fence" that prevent us from re-arranging/re-distributing the a's, and it makes it so that when we pump we only pump the leading a's.

   (c) $L = \{0^n 1^m 0^n | m, n \geq 0\}$
   AFSOC L is regular with pumping length p. Take $w = 0^p 1 0^p = \underbrace{0 \ldots 0}_{p} 1 \underbrace{0 \ldots 0}_{p}$. Clearly $w \in L$ and $|w| \geq p$, so w should be pumpable. Split up $w = xyz$ where $|xy| \leq p$ and $|y| > 0$. Since $|xy| \leq p$, y contains only <u>leading</u> 0s. When we pump up y, there will be more leading 0s than trailing 0s. The new string will not be in the language, which is a contradiction of the pumping lemma. Thus, L is not regular.

3. (a) $L = \{w | w \in \{0,1\}^*, w \text{ is a palindrome}\}$

AFSOC L is regular with pumping length p. Take $w = 0^p10^p = \underbrace{0\ldots0}_{p}1\underbrace{0\ldots0}_{p}$. Clearly $w \in L$ and $|w| \geq p$, so w should be pumpable. Split up $w = xyz$ where $|xy| \leq p$ and $|y| > 0$. Since $|xy| \leq p$, y contains only leading 0s. When we pump up y, there will be more leading 0s than trailing 0s; thus it won't read the same forwards and backwards. The new string will not be in the language, which is a contradiction of the pumping lemma. Thus, L is not regular.

(b) $L = \{w | w \in \{0,1\}^*, w \text{ is not a palindrome}\}$

AFSOC L is regular with pumping length p. Following the hint, take $s = 0^p10^{p!+p} = \underbrace{0\ldots0}_{p}1\underbrace{0\ldots0}_{\text{lots of 0s!}}$, which is not a palindrome. Clearly $w \in L$ and $|w| \geq p$, w should be pumpable.

Split up $w = xyz$ where $|xy| \leq p$ and $|y| > 0$. Since $|xy| \leq p$, y must contain only leading 0s. We now want to figure out a way to pump the string so the leading and trailing 0s *are* equal.

Let's let i be the number of 0s contained in x and let j be the number of 0s contained in y. So the string looks like this

$$\underbrace{0\ldots0}_{|x|=i}\underbrace{0\ldots0}_{|y|=j}\overbrace{\underbrace{0\ldots0}_{p-i-j}10^{p!+p}}^{z}$$

From this, we see that xy^nz contains $i + n \times j + (p - i - j)$ leading 0s.

Consider $n = \frac{p!}{j} + 1 = \frac{1 \times 2 \times \cdots \times (p-1) \times p}{j} + 1$. Note that this is an integer because $1 \leq j \leq p$.

What happens when we pump up n times? As noted above, the number of leading 0s will be

$$\begin{aligned} i + n \times j + (p - i - j) &= i + \left(\frac{p!}{j} + 1\right) \times j + (p - i - j) \\ &= i + p! + j + (p - i - j) = p! + p \end{aligned}$$

Lo and behold, this is also the number trailing 0s (which were not affected by pumping)! Thus, we've turned our string into a palindrome; $xy^nz \notin L$, which is a contradiction of the pumping lemma. Thus, L is not regular.

(c) $L = \{w | w \in \{0,1\}^*, w \text{ is not a palindrome}\}$

AFSOC L is regular. Then L^c is regular, because regular languages are closed under complement. However, we proved in part (a) that $L^c = \{w | w \in \{0,1\}^*, w \text{ is a palindrome}\}$ is not regular. This is a contradiction, and we conclude that L is not regular.

4. PRIMES $= \{w | w \in \Sigma^*, \text{ the length of w is a prime number}\}$

AFSOC L is regular with pumping length p. Let q be a prime number that is greater than p. Take $w = 1^q = \underbrace{1\ldots1}_{q \text{ is prime}}$. Clearly $w \in L$ and $|w| \geq p$, so w should be pumpable.

Split up $w = xyz$, with $|y| > 0$ (and also $|xy| \leq p$, although that won't be too important this time.) Let i be the number of 1s contained in y. So the string looks like this

$$\overbrace{\underbrace{1\ldots1}_{x}\underbrace{1\ldots1}_{|y|=i}\underbrace{1\ldots1}_{z}}^{|x|+|z|=q-i}$$

From this we see that xy^nz contains $i \times n + (q - i)$ total 1s.

Let's let $n = (q+1)$. Then the length of xy^nz will be

$$\begin{aligned} i \times (q+1) + (q-i) &= q \cdot i + i + q - i \\ &= q \cdot i + q = q \cdot (i+1). \end{aligned}$$

Both q and $i+1$ are greater than 2, so $q \cdot (i+1)$ cannot be a prime number. Thus, $xy^nz \notin L$, which is a contradiction of the pumping lemma. We conclude that L is not regular.