Mathematical notation, sets, and proof techniques

Arjun Chandrasekhar

Greek symbols

- Σ, σ - "sigma"
- Γ, γ - "gamma"
- δ - "delta"
- ϵ, ε - "epsilon"

Logical symbols

- $\Rightarrow$ - “implies”, “if ... then ...”
- $\Leftrightarrow$ - “equivalent”, “if and only if”
- $\vee$ - “OR”
- $\wedge$ - “AND”
- $\neg$ - “NOT”

Set notation

- $\in$ - "Set inclusion", "is an element of"
  - $1 \in \{1, 2, 3, 4\}$
- $\notin$ - "Set exclusion", "is not an element of"
  - $10 \notin \{1, 2, 3, 4\}$
- $A^c, \overline{A}$ - "complement of A"
  - $\{x | x \notin A\}$
- $A \cup B$ - "Union", "A or B"
  - $\{x | x \in A \vee x \in B\}$
- $A \cap B$ - "Intersection", "A and B"
  - $\{x | x \in A \wedge x \in B\}$
- $A \backslash B$ - "Set difference", "A but not B"
  - $\{x | x \in A, x \notin B\}$

De Morgan's Laws

- $(A \cup B)^c = (A^c \cap B^c)$
- $(A \cap B)^c = (A^c \cup B^c)$
- **Note:** $(A^c)^c = A$

Set closure

- Let S be a set
- Let $\Phi(x_1, x_2, \ldots, x_n)$ be a function/operation
- Suppose $x_1, \ldots x_n \in S \Rightarrow \Phi(x_1, \ldots x_n) \in S$
- We say S is **closed under** Φ
- "If we start with elements of S and apply the operation Φ, the result is still an element of S"

Set Closure

- The integers are closed under addition/+
  - If you start with two integers and add the together, you get an integer
- The integers are closed under subtraction/-
  - If you start with two integers and subtract them, you get an integer
- The positive integers are not closed under subtraction/-
  - If you start with two positive integers and subtract them, you might get a negative number

Set closure

Which of the following operations are the positive integers closed under?

A. Negation
B. Reversing the digits
C. Multiplication
D. Division

Set closure

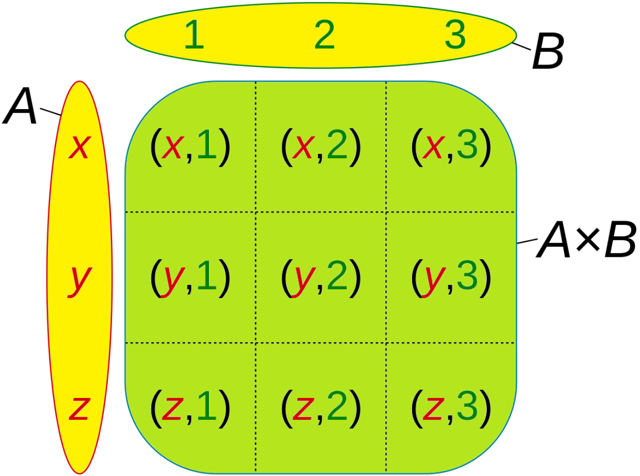
Which of the following operations are the positive integers closed under?

A. Negation
B. Reversing the digits ✓
C. Multiplication ✓
D. Division

Cartesian product

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Cartesian product



Cartesian product

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

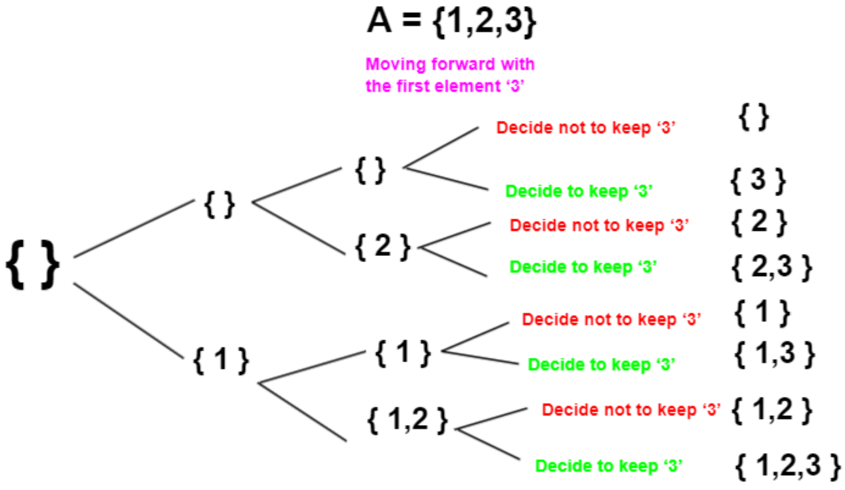
Let $A = \{1, 2\}$, $B = \{\text{red, blue}\}$, $C = \{1502\}$

- What is $A \times B$?
  - $\{(1, \text{red}), (1, \text{blue}), (2, \text{red}), (2, \text{blue})\}$
- What is $A \times A$?
  - $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$
- What is $C \times C \times B$?
  - $\{(1502, 1502, \text{red}), (1502, 1502, \text{blue})\}$

Power set

- Let S be a set
- The **power set** $\mathcal{P}(S)$ is the set of all subsets of S

Power set


A = {1,2,3}
Moving forward with
the first element '3'
{ }
{ }
{ }
{ 2 }
{ 1 }
{ 1 }
{ 1,2 }
Decide not to keep '3'
Decide to keep '3'
Decide not to keep '3'
Decide to keep '3'
Decide not to keep '3'
Decide to keep '3'
Decide not to keep '3'
Decide to keep '3'
{ }
{ 3 }
{ 2 }
{ 2,3 }
{ 1 }
{ 1,3 }
{ 1,2 }
{ 1,2,3 }

Power set

- Let S be a set
- The **power set** $\mathcal{P}(S)$ is the set of all subsets of S
- What is $\mathcal{P}(\{0,1,2\})$?
  - $\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}\}$
- What is $\mathcal{P}(\{\text{even}, \text{odd}\})$?
  - $\mathcal{P}(\{\text{even}, \text{odd}\}) = \{\emptyset, \{\text{even}\}, \{\text{odd}\}, \{\text{even}, \text{odd}\}\}$
- What is $\mathcal{P}(\{(0,1,2,3,4,5)\})$?
  - $\mathcal{P}(\{(0,1,2,3,4,5)\}) = \{\emptyset, \{(0,1,2,3,4,5)\}\}$

Proof by Induction

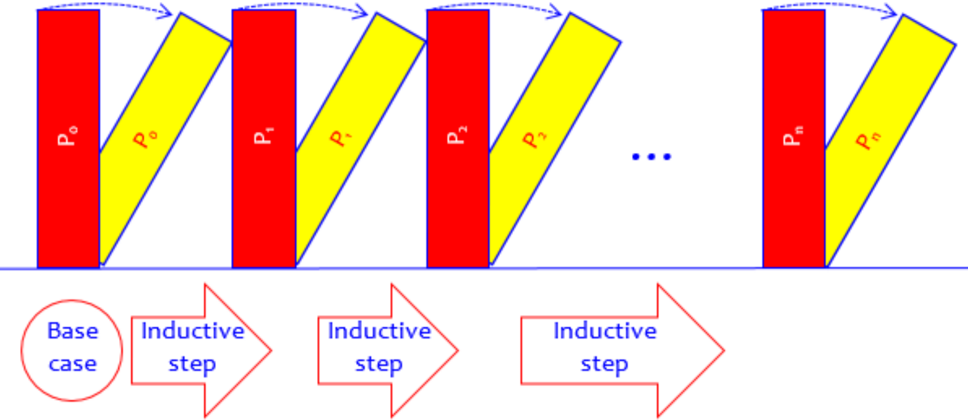
- Imagine several dominoes lined up
  - Suppose we figure out how to knock over the first domino
  - Suppose we line up the dominoes so that if one domino falls, it is guaranteed to knock over the first domino
  - What happens to the rest?

Proof by Induction

Proof by Induction

- **Base Case**: prove the claim for the simplest possible example
- **Inductive Case**: prove that if the claim is true for $n = k$, then the claim must be true for $n = k + 1$
- Conclude the claim is true for all N
  - $N = 0 \implies N = 1$
  - $N = 1 \implies N = 2$
  - $\ldots$

Proof by Induction



Proof by induction: Gauss's formula

Let's prove Gauss's formula: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- **Inductive case:** Assume that the formula holds true for n, i.e. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
- We want to show the formula is true for $n+1$, i.e. $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

$$\sum_{i=1}^{n+1} i = n+1+\sum_{i=1}^{n} i = n+1+\frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Proof by induction example

Let's prove that for $n \geq 4$, $2^n < n!$

- **Base Case:** Let $n = 4$

$$2^4 = 16 < 24 = 4!$$

Proof by Induction Example

Let's prove that for $n \geq 4$, $2^n < n!$

- **Inductive Case:** Assume the formula holds for n, i.e. $2^n < n!$
- Let's prove it for $n + 1$, i.e. $2^{n+1} < (n+1)!$

  $2^{n+1} = 2 \times 2^n < (n+1) \times n! = (n+1)!$

AFSOC

Assume
For
Sake
Of
Contradiction

(h/t Stephen Worlow)

Proof by contradiction

Suppose we want to prove that x is true, and we know that y is false. We prove x as follows:

1. Assume for sake of contradiction (AFSOC) that x is false
2. Show that $\neg x \implies y$ (which we know is false)
3. Conclude that $\neg x$ is false, and x is true

Proof

Supp ve
know

1.)
   t
2. false)
3.

Proof by Contradiction Example

Theorem: My third grade teacher did not witness Lincoln's assassination.

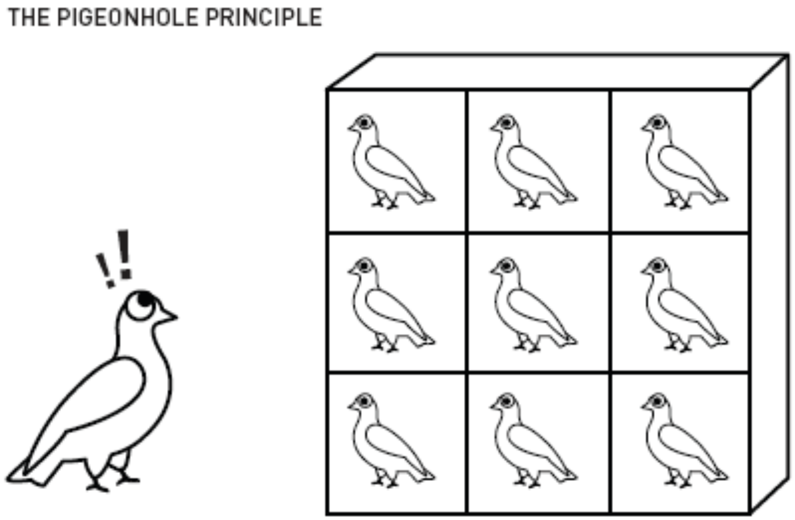
- AFSOC that my third grade teacher witnessed Lincoln's assassination.
- Then my third grade teacher would have been at least 135 years old when I was her student
  - No human has ever lived to be that old
- Our original assumption leads to a logical contradiction.
- Thus, we conclude that she did not witness Lincoln's assassination.

Proof by Contradiction Example

Proposition: There are infinitely many primes.

- AFSOC there are finitely many primes $p_1, p_2, \ldots, p_n$
- Let $N = p_1 \cdot p_2 \cdot \ldots p_{n-1} \cdot p_n + 1$
- None of $p_1, p_2, \ldots p_n$ divides N
- N has no prime factors, which is a contradiction
- We conclude that our original assumption was incorrect.

The Pigeonhole Principle



The Pigeonhole Principle

- The **pigeonhole principle** states that if there are m pigeonholes and $n > m$ pigeons, there must be a pigeonhole with multiple pigeons
  - Formally, there does not exist an injective function whose co-domain is smaller than its domain
- If you have three gloves, at least two of them must fit the same hand
- A room of 367 people must include a shared birthday

The Pigeonhole Principle

Proposition: Let G be a directed graph in which every vertex has at least one outgoing edge. Prove that G has a directed cycle.

- Perform a random walk. v_1 can be any vertex, and v_{i+1} can be outbound neighbor of v_i
  - It is always possible to continue the walk, since every vertex has at least one outgoing edge
- Let n be the number of vertices in G. Look at $v_1, v_2, \ldots, v_{n+1}$
- By the pigeonhole principle, one of the first $n+1$ vertices must be a repeat - which gives us our cycle