Mathematical notation, sets, and proof techniques

Arjun Chandrasekhar

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- **Note:** $(A^c)^c = A$

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- "If we start with elements of S and apply the operation Φ, the result is still an element of S"

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  - If you start with two positive integers and subtract them, you might get a negative number

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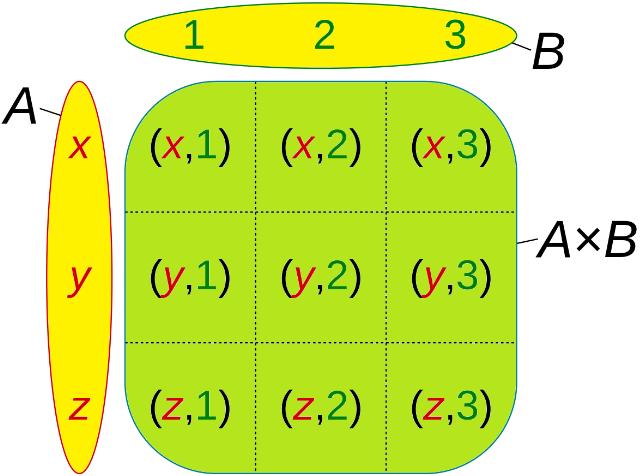

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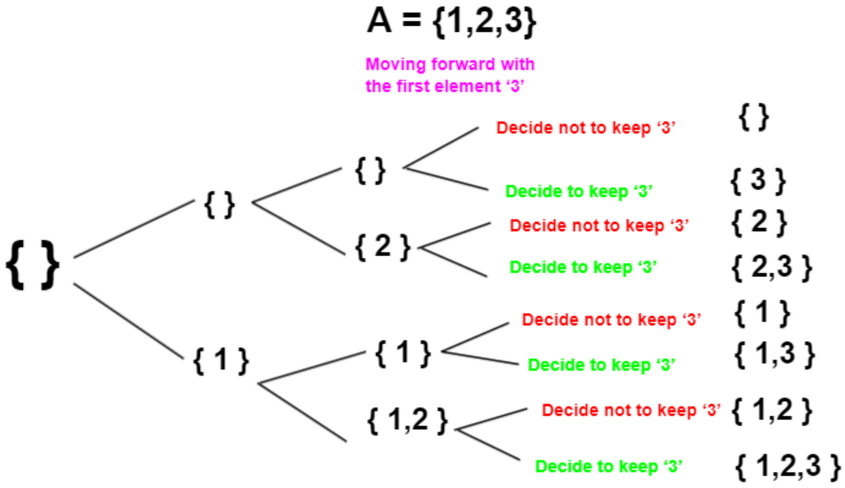
A = {1,2,3}
Moving forward with
the first element '3'
{ }
{ }
{ }
{ 2 }
{ 1 }
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{ 1,2 }
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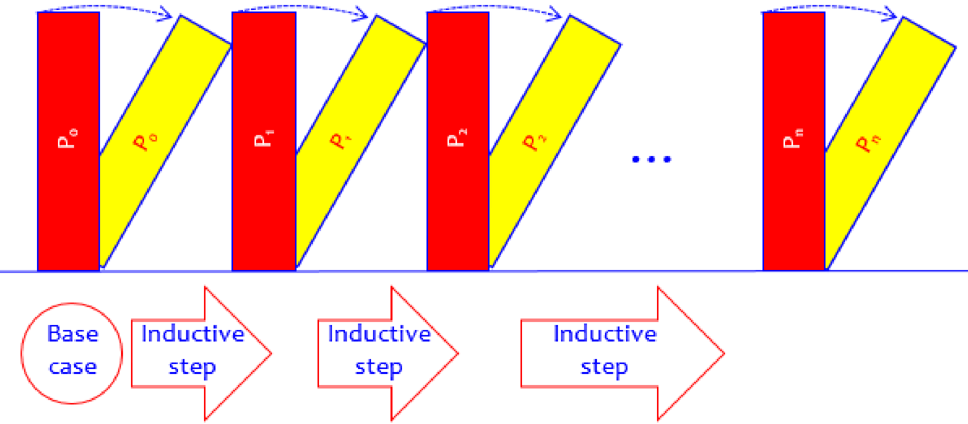

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$$2^4 = 16 < 24 = 4!$$

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  $2^{n+1} = 2 \times 2^n < (n+1) \times n! = (n+1)!$

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(h/t Stephen Worlow)

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3. Conclude that $\neg x$ is false, and x is true

Proof

Supp ... we
know

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   t
2. S ... false)
3. C

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- Thus, we conclude that she did not witness Lincoln's assassination.

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- N has no prime factors, which is a contradiction
- We conclude that our original assumption was incorrect.

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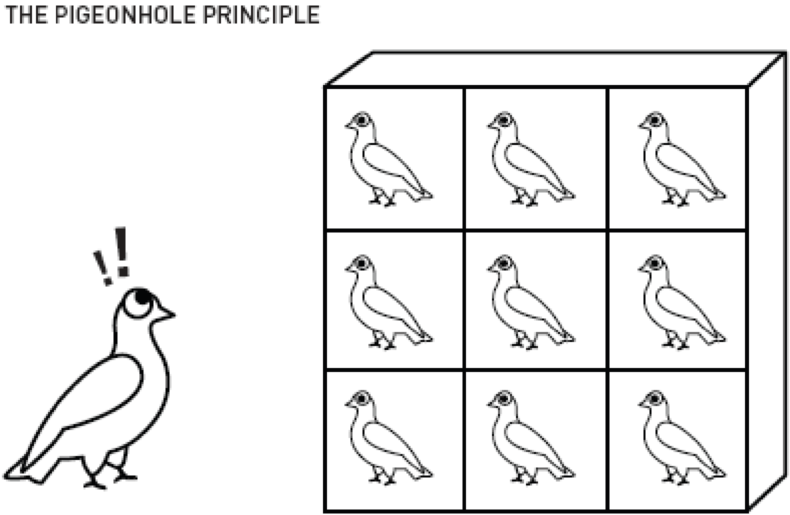

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  - Formally, there does not exist an injective function whose co-domain is smaller than its domain

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- A room of 367 people must include a shared birthday

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- By the pigeonhole principle, one of the first $n + 1$ vertices must be a repeat - which gives us our cycle