Theory of Computation
Poly-time reductions,
NP-completeness

The million dollar question

"What is the largest group of Facebook users that are all connected to each other"

- Can you write an *efficient* algorithm to answer this question?
- **Can you prove that no efficient algorithm exists for this problem?**

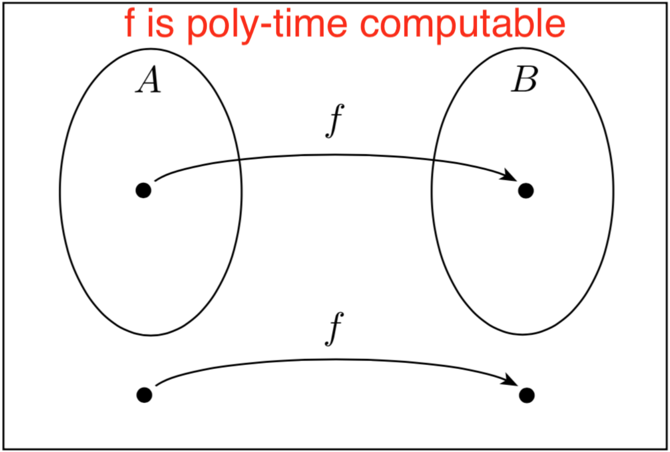
Poly-time computable functions

- **Recall:** A function $f : \Sigma^* \to \Sigma^*$ is **computable** if there is a Turing machine M that *computes* it
  - If we start with w on the tape, M will halt leave $f(w)$ on the tape
- **Def:** a computable function f is **poly-time computable** if M runs in polynomial time

Poly-time reductions

- **Recall:** We say $A \leq_M B$ if there is a computable function $f : \Sigma^* \to \Sigma^*$ such that $w \in A \Leftrightarrow f(w) \in B$
  - "YES maps to YES"
  - "NO maps to NO"
- **Def:** We say A is **poly-time reducible** to B (denoted $A \leq_{\text{poly}} B$) if the reduction f is poly-time computable
- Informally, it means that we can "convert" an instance of A to an instance of B in polynomial time

Poly-time reductions

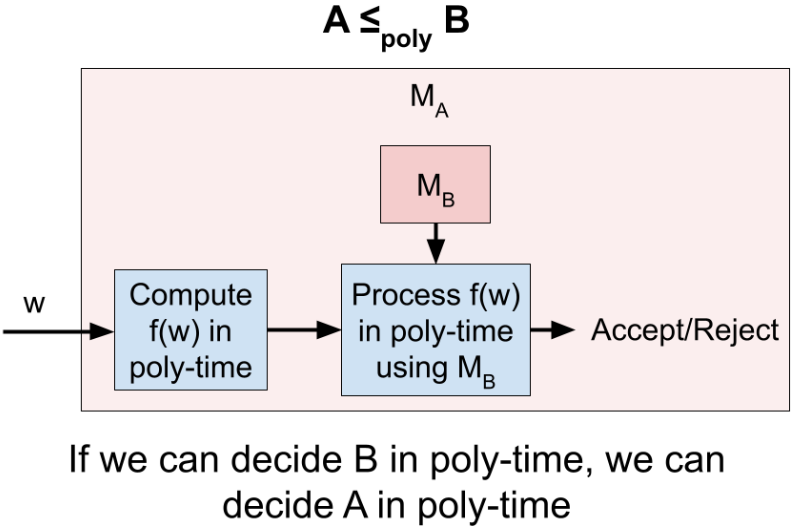


Implications of poly-time reducibility

Theorem: If $B \in \mathrm{P}$ and $A \leq_{\mathrm{poly}} B$, then $A \in \mathrm{P}$

- Since $B \in \mathrm{P}$, there is a machine M_B that decides B in poly-time
- Since $A \leq_{\mathrm{poly}} B$ there is a poly-time computable function f such that $w \in A \Leftrightarrow f(w) \in B$
- Create the following machine poly-time M_A to decide A
  1. Compute $f(w)$ (poly-time)
  2. Run M_B on $f(w)$ (poly-time)
  3. If M_B accepts $f(w)$ then M_A accepts w. Otherwise, M_A rejects w.

Implications of polytime-reducibility

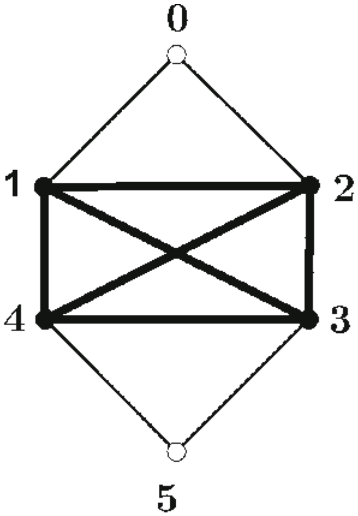


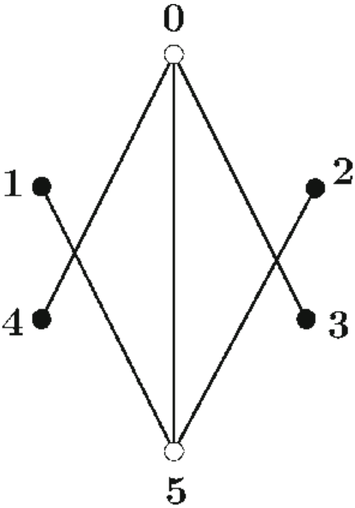
IND-SET $\leq_{\text{poly}}$ CLIQUE

We reduce from IND-SET to CLIQUE as follows:

1. **Input:** A graph G with V vertices and E edges, and an integer k
2. Create the **complement graph** $\overline{G}$ by reversing all of the edges in G
3. Check if $\overline{G}$ has a clique of size k. If so, accept $\langle G, k \rangle$; otherwise reject

IND-SET $\leq_{\text{poly}}$ CLIQUE





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Poly-time: $O(E)$ to construct $\overline{G}$

IND-SET $\leq_{\text{poly}}$ CLIQUE

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"YES maps to YES": If G has a k-independent set, then those same vertices will all be connected in $\overline{G}$

IND-SET $\leq_{\text{poly}}$ CLIQUE

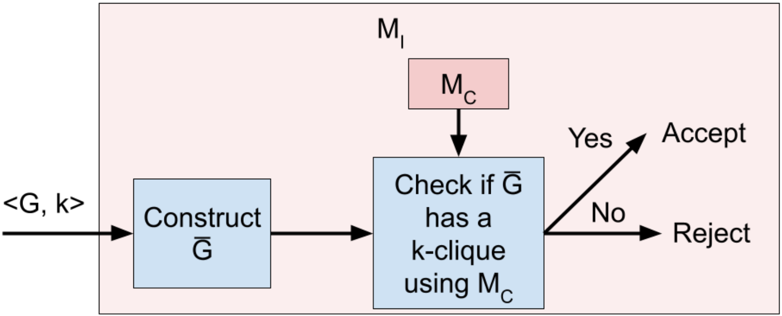
We reduce from IND-SET to CLIQUE as follows:

1. **Input:** A graph G with V vertices and E edges, and an integer k
2. Create the **complement graph** $\overline{G}$ by reversing all of the edges in G
3. Check if $\overline{G}$ has a clique of size k. If so, accept $\langle G, k \rangle$; otherwise reject

"NO maps to NO": If G doesn't have a k-independent set, then every set of k vertices has at least one edge. Those same vertices will be missing an edge in $\overline{G}$

IND-SET $\leq_{\text{poly}}$ CLIQUE

IND-SET $\leq_{poly}$ CLIQUE



If we can decide CLIQUE in poly-time, we can decide IND-SET in poly-time

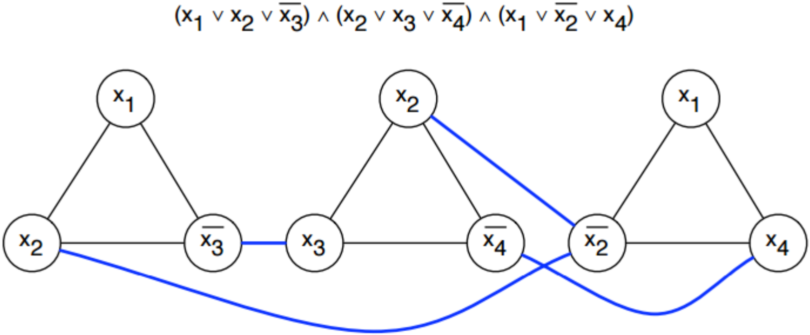
3-SAT $\leq_{\text{poly}}$ IND-SET

We reduce from IND-SET to CLIQUE as follows:

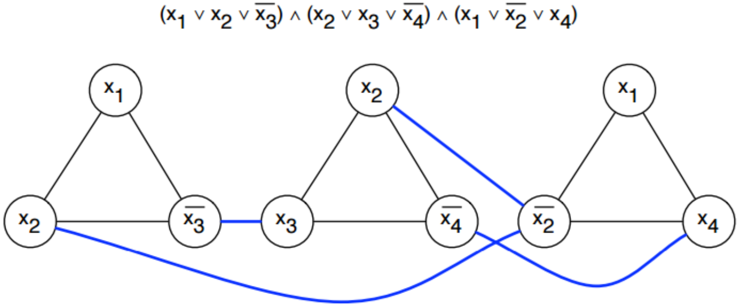
1. **Input:** a 3-CNF formula with n variables and m clauses
2. Create a graph G
3. For each clause $(x \vee y \vee z)$, create three nodes x, y, z and connect them to form a "triangle"
4. If there are nodes x and $\neg x$, connect them with an edge
5. Check if there is an independent set of size m

3-SAT $\leq_{\text{poly}}$ IND-SET

We reduce from IND-SET to CLIQUE as follows:

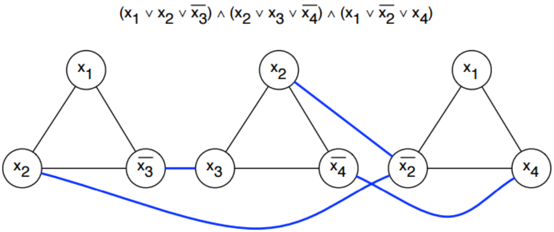


3-SAT $\leq_{\text{poly}}$ IND-SET: poly-time



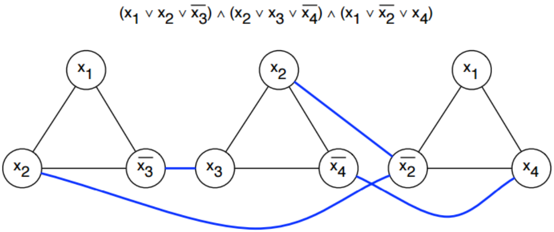
- $O(m)$ vertices
- $O(m) + O(n^2)$ edges
- $O(m) + O(n^2) =$ poly-time

3-SAT $\leq_{\text{poly}}$ IND-SET: yes $\rightarrow$ yes



- Suppose F has a satisfying assignment
- For each "triangle", pick one of the TRUE vertices to be in the independent set
  - Every clause has at least one true variable
  - Variables from different clauses are not connected
  - Truth assignment will not let us pick x and $\neg x$
- m clauses $\rightarrow$ m triangles $\rightarrow$ m-independent set

3-SAT $\leq_{\text{poly}}$ IND-SET: no $\rightarrow$ no



Show the contrapositive: <u>yes $\leftarrow$ yes</u>

- Suppose G has a an independent set of size m
- Set the variables that are part of the independent set to be TRUE
  - There must be one vertex from each "triangle" in the set, so every clause will be satisfied
  - x and $\neg x$ are connected, so our independent set will not include a contradictory assignment

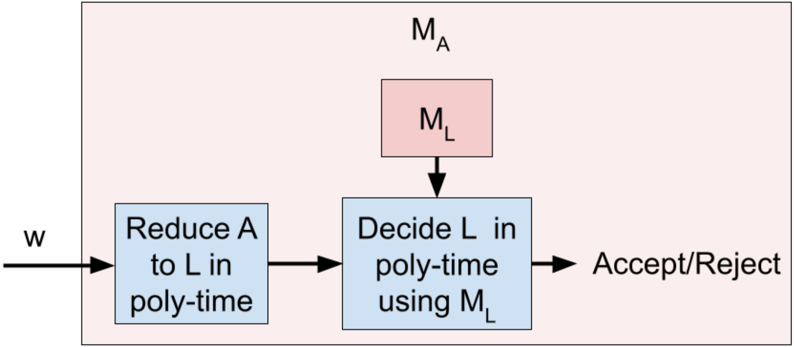
NP-completeness

- **Def:** A language L is **NP-Hard** if *every* language in NP is poly-time reducible to L
  - $A \in \mathrm{NP} \implies A \leq_{\mathrm{poly}} L$
- **Def:** L is NP-complete if:
  1. $L \in \mathrm{NP}$
  2. L is NP-Hard
- L is the "hardest" or "most expressive" problem in NP

NP-completeness

L is NP-Hard

A ∈ NP



If we can decide L in poly-time, we can decide *every* NP language in poly-time!

3-SAT is NP-complete

Cook-Levin theorem: $\text{CIRCUIT} - \text{SAT}$ is NP-complete

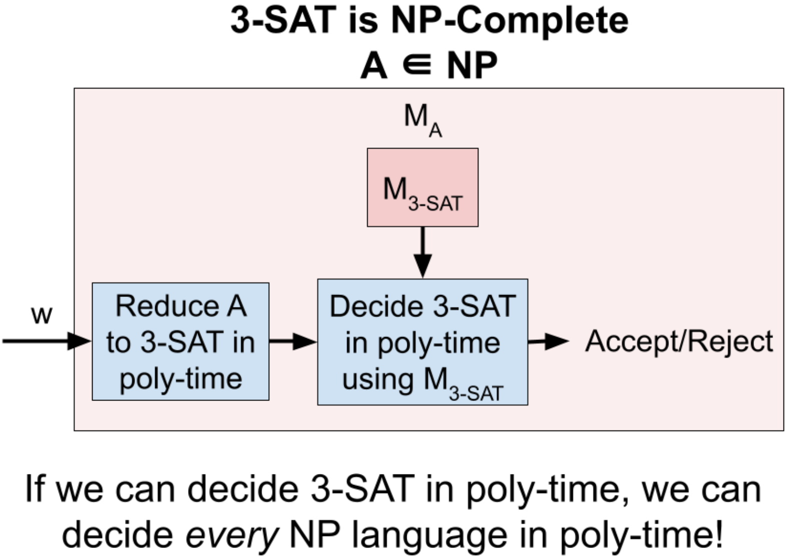
- Like 3-SAT, but we can use any combination of $\neg, \vee, \wedge$
- **Proof idea:** create a boolean circuit that checks if the input string eventually leads to an accepting computation history

Karp's theorem: 3-SAT is NP-complete

- Every boolean circuit can be converted to a 3-CNF circuit

See Sipser for full proof

3-SAT is NP-complete



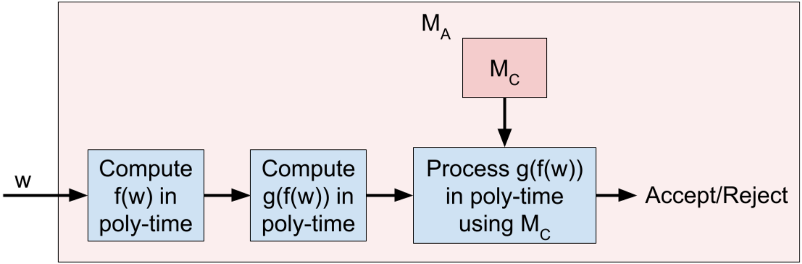
Transitivity of $\leq_{\text{poly}}$

Proposition: If $A \leq_{\text{poly}} B$ and $B \leq_{\text{poly}} C$, then $A \leq_{\text{poly}} C$

- There exists a poly-time computable function f such that $w \in A \Leftrightarrow f(w) \in B$
- There exists a poly-time computable function g such that $w \in B \Leftrightarrow g(w) \in C$
- $w \in A \Leftrightarrow f(w) \in B \Leftrightarrow g(f(w)) \in C$
- $g \circ f$ is a poly-time reduction from A to C!

Transitivity of $\leq_{\text{poly}}$

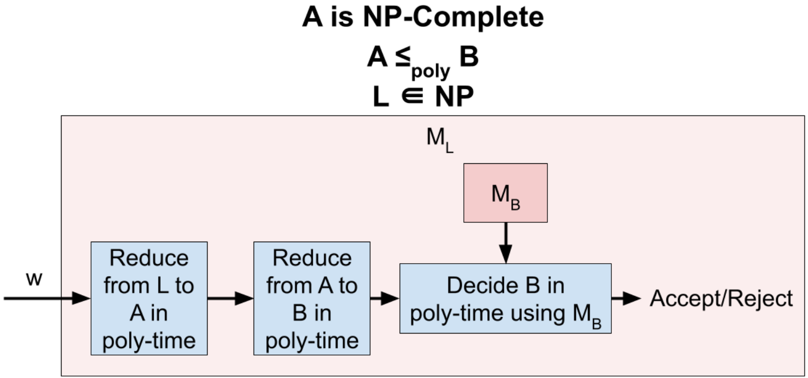
A $\leq_{poly}$ B
B $\leq_{poly}$ C



If we can decide C in poly-time,
we can decide A in poly-time

Transitivity of NP-Completeness

Corollary: If A is NP-complete, and $A \leq_{\text{poly}} B$, then B is NP-complete



If we can decide B in poly-time, we can decide *any* language in NP in poly-time!

Implications of 3-SAT NP-Completeness

- **We can use 3-SAT to prove that other languages are NP-complete!**
  - If we can show that $3\text{-SAT} \leq_{\text{poly}} L$, it follows that L is also complete!
- And we can use those other languages to show that even more languages are NP-complete

Implications of 3-SAT NP-Completeness

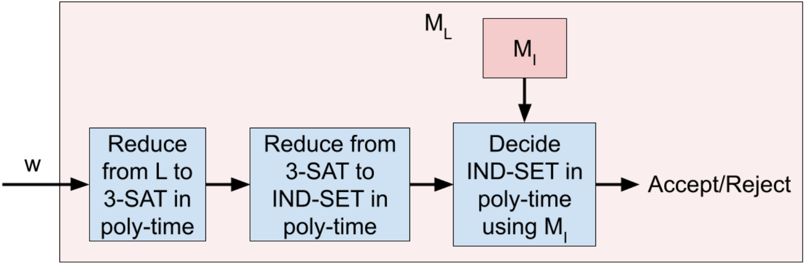
IND-SET is NP-Complete

- 3-SAT is known to be NP-complete
- We proved that 3-SAT $\leq_{\text{poly}}$ IND-SET
- Thus, IND-SET is NP-complete

3-SAT is NP-Complete

3-SAT $\leq_{\text{poly}}$ IND-SET

L ∈ NP



If we can decide IND-SET in poly-time, we can decide *any* language in NP in poly-time!

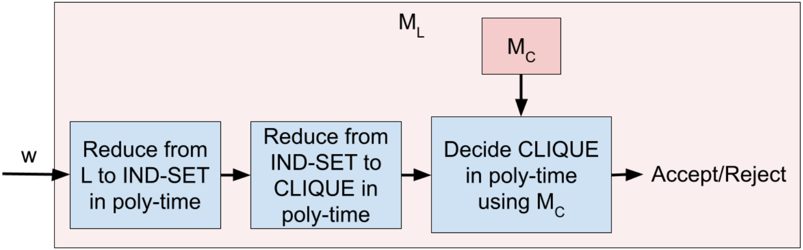
CLIQUE is NP-Complete

- IND-SET is known to be NP-Complete
- We proved that IND-SET $\leq_{\text{poly}}$ CLIQUE
- Thus, CLIQUE is NP-Complete

IND-SET is NP-Complete

IND-SET $\leq_{\text{poly}}$ CLIQUE

L $\in$ NP



If we can decide CLIQUE in poly-time, we can decide *any* language in NP in poly-time!

SUBSET-SUM is NP-Complete

Proof: Reduce from 3-SAT

1. We will create a number for each variable x_i and its negation
   - The digits of the number correspond to which clauses that variable can satisfy
2. We will set the target sum such that it can only be reached through a satisfying assignment
   - To reach the target, each clause needs to have at least of its true
3. We will set the desired sum such that each clause needs to satisfied

3-SAT $\leq_{\text{poly}}$ SUBSET-SUM: variables

- We want our numbers to correspond to assigning each variable to TRUE or FALSE
- For each variable x_i, we will create two numbers: x_i^{TRUE} and x_i^{FALSE}
- We will design our desired total so that exactly one of these two numbers must be picked

3-SAT $\leq_{\text{poly}}$ SUBSET-SUM: variables

$$F = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor \neg x_3) \land \ldots \land (x_2 \lor x_3)$$

n digits (1 per unique variable)

2n numbers (2 per unique variable)

x_1^{TRUE}	= 1	0	0	0	…	0
x_1^{FALSE}	= 1	0	0	0	…	0
x_2^{TRUE}	= 0	1	0	0	…	0
x_2^{FALSE}	= 0	1	0	0	…	0
x_3^{TRUE}	= 0	0	1	0	…	0
x_3^{FALSE}	= 0	0	1	0	…	0
…						
x_n^{TRUE}	= 0	0	0	0	…	1
x_n^{FALSE}	= 0	0	0	0	…	1
B	**= 1**	**1**	**1**	**1**	**…**	**1**

Each variable must be TRUE or FALSE

3-SAT $\leq_{\text{poly}}$ SUBSET-SUM: clauses

- We want our numbers to correspond to satisfying certain clauses
- For each number, we will add an extra digit for each clause
  - The extra digits signifiy which variables satisfy which clauses
- We will design our desired total so that (at least) one variable must be picked for each clause

3-SAT $\leq_{\text{poly}}$ SUBSET-SUM: clauses

$F = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge \ldots \wedge (x_2 \vee x_3)$

m digits (1 per clause)

							c_1	c_2		c_m
x_1^{TRUE} =	1	0	0	0	…	0	1	0	…	0
x_1^{FALSE} =	1	0	0	0	…	0	0	1	…	0
x_2^{TRUE} =	0	1	0	0	…	0	1	1	…	1
x_2^{FALSE} =	0	1	0	0	…	0	0	0	…	0
x_3^{TRUE} =	0	0	1	0	…	0	0	0	…	1
x_3^{FALSE} =	0	0	1	0	…	0	1	1	…	0
…							…			
x_n^{TRUE} =	0	0	0	0	…	1	0	0	…	0
x_n^{FALSE} =	0	0	0	0	…	1	0	0	…	0
B =	1	1	1	1	…	1	?	?	…	?

Which variables satisfy which clauses?

How do we ensure that each clause is satisfied?

3-SAT $\leq_{\text{poly}}$ SUBSET-SUM: clauses

- How do we design our target B so that each clause must be satisfied?
- **Attempt 1:** Include a 1 digit for each clause
  - **Problem:** What if a clause has more than one TRUE variable?
- **Attempt 2:** Include a 3 digit for each clause
  - **Problem:** A satisfied clause might have only 1 or 2 TRUE variables
- How do we represent "between 1 and 3" when subset sum requires an exact total?
- We will introduce **filler numbers**

3-SAT $\leq_{\text{poly}}$ SUBSET-SUM: fillers

- For each clause, introduce two *fillers*
- From a given clause, if at least one variable is TRUE, we can use up to two fillers to bring the total for that clause to 3
- If all variables in a clause are FALSE, then that clause will never add up to 3 (even with the fillers)

3-SAT $\leq_{poly}$ SUBSET-SUM: fillers

F = $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge \ldots \wedge (x_2 \vee x_3)$

x_1^{TRUE}	= 1	0	0	0	…	0	1	0	…	0
x_1^{FALSE}	= 1	0	0	0	…	0	0	1	…	0
x_2^{TRUE}	= 0	1	0	0	…	0	1	1	…	1
x_2^{FALSE}	= 0	1	0	0	…	0	0	0	…	0
x_3^{TRUE}	= 0	0	1	0	…	0	0	0	…	1
x_3^{FALSE}	= 0	0	1	0	…	0	1	1	…	0
…							…			
x_n^{TRUE}	= 0	0	0	0	…	1	0	0	…	0
x_n^{FALSE}	= 0	0	0	0	…	1	0	0	…	0
$fill_{11}$	= 0	0	0	0	…	0	1	0	…	0
$fill_{12}$	= 0	0	0	0	…	0	1	0	…	0
$fill_{21}$	= 0	0	0	0	…	0	0	1	…	0
$fill_{22}$	= 0	0	0	0	…	0	0	1	…	0
…										
$fill_{m1}$	= 0	0	0	0	…	0	0	0	…	1
$fill_{m2}$	= 0	0	0	0	…	0	0	0	…	1
B	**= 1**	**1**	**1**	**1**	**…**	**1**	**3**	**3**	**…**	**3**

Fillers don't affect variable truth assignments

Can use up to 2 fillers per clause

Need ≥ 1 TRUE variable + ≤ 2 fillers

3-SAT $\leq_{poly}$ SUBSET-SUM

$$F = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge \ldots \wedge (x_2 \vee x_3)$$

x_1^{TRUE}	= 1	0	0	0	…	0	1	0	…	0
x_1^{FALSE}	= 1	0	0	0	…	0	0	1	…	0
x_2^{TRUE}	= 0	1	0	0	…	0	1	1	…	1
x_2^{FALSE}	= 0	1	0	0	…	0	0	0	…	0
x_3^{TRUE}	= 0	0	1	0	…	0	0	0	…	1
x_3^{FALSE}	= 0	0	1	0	…	0	1	1	…	0
…							…			
x_n^{TRUE}	= 0	0	0	0	…	1	0	0	…	0
x_n^{FALSE}	= 0	0	0	0	…	1	0	0	…	0
$fill_{11}$	= 0	0	0	0	…	0	1	0	…	0
$fill_{12}$	= 0	0	0	0	…	0	1	0	…	0
$fill_{21}$	= 0	0	0	0	…	0	0	1	…	0
$fill_{22}$	= 0	0	0	0	…	0	0	1	…	0
…										
$fill_{m1}$	= 0	0	0	0	…	0	0	0	…	1
$fill_{m2}$	= 0	0	0	0	…	0	0	0	…	1
B	**= 1**	**1**	**1**	**1**	**…**	**1**	**3**	**3**	**…**	**3**

3-SAT $\leq_{\text{poly}}$ SUBSET-SUM: poly-time

- $O(n)$ "variable" numbers
- $O(m)$ "filler" numbers
- Each number has $O(n+m)$ base-10 digits
- $(O(n) + O(m)) \cdot O(n+m) =$ poly-time
- **Note:** The length of the numbers would be exponential if we used a unary encoding
  - If we could find a poly-time reduction that uses unary, we would have proven that $\mathrm{P} = \mathrm{NP}$

3-SAT $\leq_{\text{poly}}$ SUBSET-SUM: yes $\rightarrow$ yes

"YES maps to YES":

- Suppose F has a satisfying assignment
- If x_i is assigned TRUE, include x_i^{TRUE} in our subset. Otherwise, include x_i^{FALSE}
- A variable and its negation will never both be assigned TRUE, so we have a 1 in the first n positions of B
- Each clause is satisfied, so we have at least 1 in the last m positions of B
- Can use up to 2 fillers to get a 3 in the last m positions of B

3-SAT $\leq_{\text{poly}}$ SUBSET-SUM: no $\rightarrow$ no

"<u>NO maps to NO</u>":

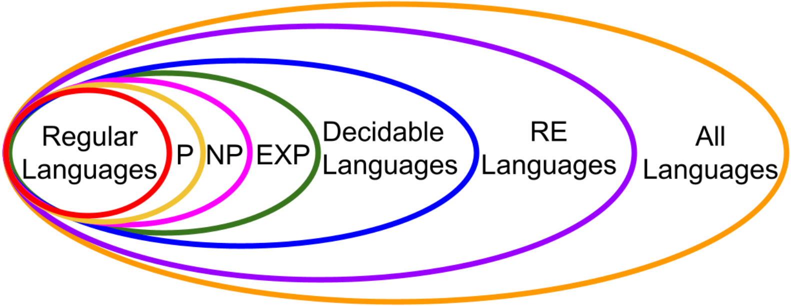
- Suppose F is unsatisfiable
- Every satisfying assignment will leave at least one clause unsatisfied
- One of the last m digits of our subset will add up to at most 2
  - Without at least one TRUE variable, we don't have enough fillers to make that clause add up to 3

3-SAT $\leq_{\text{poly}}$ SUBSET-SUM: no $\rightarrow$ no

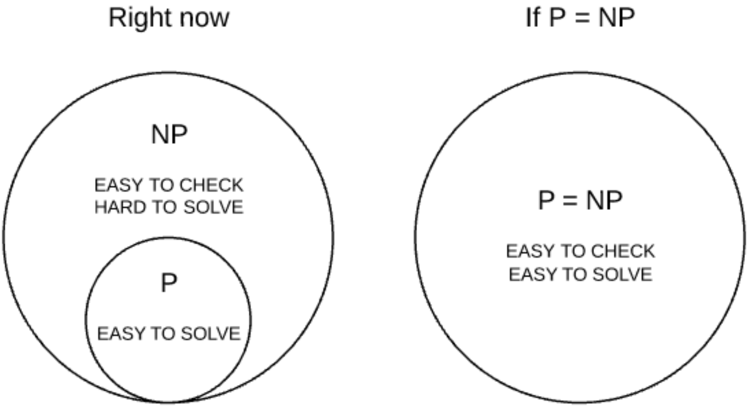
Alternately, we can prove the contrapositive: yes $\leftarrow$ yes

- Suppose there exists a subset that adds up to B
- Assign all of the variables that are part of the subset to be TRUE
- Because the first n digits of B are 1, we won't have a variable and its negation both be TRUE
- Because the last m digits of B are all 3, and there are only 2 fillers per clause, at least one variable is TRUE in each clause

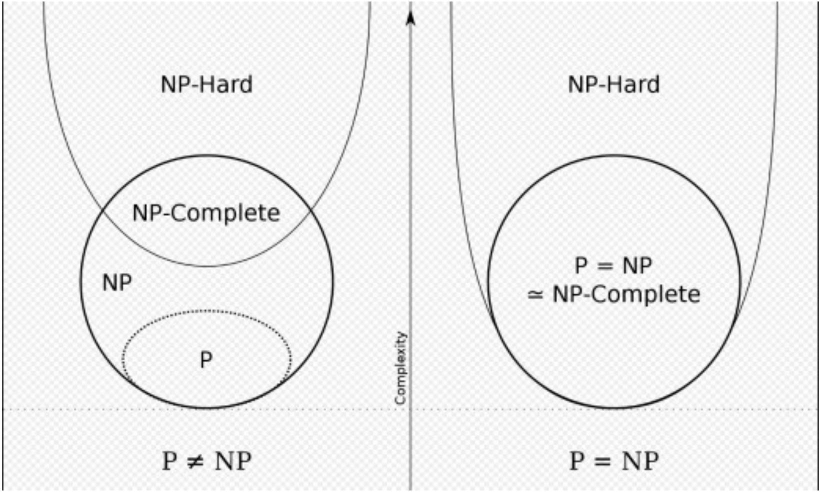
P vs. NP


Regular
Languages
P
NP
EXP
Decidable
Languages
RE
Languages
All
Languages

P vs. NP


Right now
NP
EASY TO CHECK
HARD TO SOLVE
P
EASY TO SOLVE
If P = NP
P = NP
EASY TO CHECK
EASY TO SOLVE

P vs. NP


NP-Hard
NP-Complete
NP
P
Complexity
P ≠ NP
NP-Hard
P = NP
≃ NP-Complete
P = NP

The million dollar question

Can you design an *efficient* algorithm to find the biggest clique on Facebook?

- If you can do this, then $\mathrm{P} = \mathrm{NP}$
- If you believe that $\mathrm{P} \neq \mathrm{NP}$, then this task is impossible

There is a million dollar bounty on the answer to this question!

Beyond P vs. NP: the class $\mathrm{co\text{-}NP}$

Def: The class $\mathrm{co\text{-}NP}$ is the set of languages whose complement is in NP

- $L \in \mathrm{co\text{-}NP} \Leftrightarrow L^c \in \mathrm{NP}$
- It is easy to verify if $w \notin L$
- **Example:** it is very easy to prove that a number is *not* prime (but harder to prove that it is prime)
- Some open questions:
  - Does $\mathrm{NP} = \mathrm{co\text{-}NP}$?
  - Does $P = \mathrm{NP} \cap \mathrm{co\text{-}NP}$ (similar to how decidable $=$ RE $\cap$ co-RE)?

Beyond P vs. NP: the class PSPACE

Def: The class PSPACE is the set of languages that can be decided using polynomial space/memory

- Space complexity is calculated based on how many extra tape squares are needed to process the input
- **Key insight:** Unlike time, space can be *reused*
- Some open questions:
  - Does $\mathrm{P} = \mathrm{PSPACE}$?
  - Does $\mathrm{NP} = \mathrm{PSPACE}$?
  - Does $\mathrm{PSPACE} = \mathrm{EXP}$?