Theory of Computation: The Recursion Theorem

Arjun Chandrasekhar

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- This lecture lead to one of my favorite teaching stories

Pop quiz!

For extra credit: write a non-empty program that prints out its own source code (without using any I/O operations). You have 20 minutes.

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Recall

- Certain problems are not decidable, or even recognizable
- Initially we proved this with Diagonalization
- Usually we prove this using reduction
  - If A is undecidable/unrecognizable, and $A \leq_M B$, then B is undecidable/unrecognizable
- Is there an alternative (perhaps easier) way to prove these results?

Today's goals

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- Understand how to construct a self-reproducing Turing machine

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- Describe Turing machines that analyze (and contradict) their own behavior

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- Understand how to construct a self-reproducing Turing machine
- Describe Turing machines that analyze (and contradict) their own behavior
- Use the recursion theorem for concise impossiblity proofs

Self-reproducing robots

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- Can we build a robot that builds another robot?

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- Can we build a robot that builds another robot-building robot?

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- Can we build a robot that builds another robot?
- Can we build a robot that builds another robot-building robot?
- Can we build a robot-building robot that builds an <u>identical</u> robot-building robot?

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2. A Turing machine can be programmed to obtain and then analyze its own description.

We will prove both parts by construction.

- In doing so, we will see why the simplicity of the TM model can be convenient!

Idea behind a self-reproducing programs

Print the following sentence twice, the second time in quotes
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  - On the first assignment, you read a string description $\langle D \rangle$ of a DFA and parsed it into an actual DFA object D

Review of notation

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- Can think of $\langle M \rangle$ as the source code, and M is the executable file
- Think back to the first two programming assignments
  - On the first assignment, you read a string description $\langle D \rangle$ of a DFA and parsed it into an actual DFA object D
  - On the second assignment, you read a string description $\langle N \rangle$ of an NFA, converted it into an equivalent DFA object D, and output a string description $\langle D \rangle$

Building blocks of our self-reproducing program

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Building blocks of our self-reproducing program

To construct a Turing machine that prints out its own description, we will split the program up into two sub-machines Q and $P_{\langle Q \rangle}$

- Machine Q takes an input string w and creates a machine P_w that always prints out the same string w
- Machine $P_{\langle Q \rangle}$ is a Turing machine that ignores its input and always prints out $\langle Q \rangle$, i.e. a description of the machine Q

Machines that only print one string

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- w is a constant that we decide on ahead of time
- Can a human construct this machine?
- Can we write a computer program to construct this machine?

Machines that create single-string machines

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Let $q(w) = \langle P_w \rangle$. That is, the function q takes as input a string w. It outputs the description of a Turing machine P_w that always prints out the string w.

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Let $q(w) = \langle P_w \rangle$. That is, the function q takes as input a string w. It outputs the description of a Turing machine P_w that always prints out the string w.

- Is q a Turing-computable function?
- If you understand what q is doing and convince yourself that q is computable, then the proof of the recursion theorem will be straightforward

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  - We can treat $\langle P_{\text{computability}} \rangle$ like any other pre-determined string w

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  - It's just like the last slide. We use the string "recursion" instead of the string "computability"

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- Can we construct a machine Q_{w} that prints out $\langle P_{\text{w}} \rangle$ for *any* arbitrary string w?
  - Yes!

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- Can we construct a machine Q_{w} that prints out $\langle P_{\mathrm{w}} \rangle$ for *any* arbitrary string w?
  - Yes!
  - It's just like the last two slides. Instead of the string "recursion" or "computability", we substitute our string of choice w

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  - Substitute in w for the part where P prints out its output string
  - We basically convert w from a constant to a parameter of the method Q

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Now we are ready to describe the machine SELF, a machine which prints out its own description. SELF consists of two sub-machines, $P_{\langle Q \rangle}$ and Q.

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Does that last step loop familiar?

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- But SELF also of a combination of $P_{\langle Q \rangle}$ and Q!
- SELF has reproduced its own description!

A Self-reproducing program

SELF = ⟨$P_{\langle Q\rangle}$Q⟩

Initially $P_{\langle Q\rangle}$ starts with control of the tape, with an input string w

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SELF = $\langle P_{\langle Q \rangle} Q \rangle$

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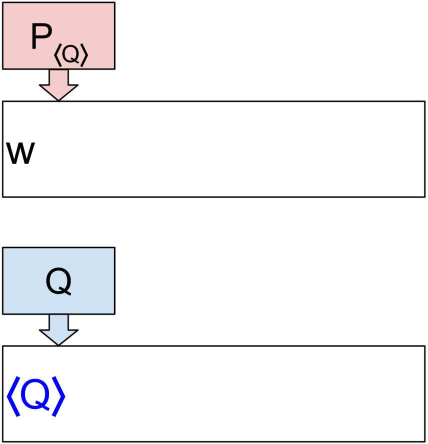

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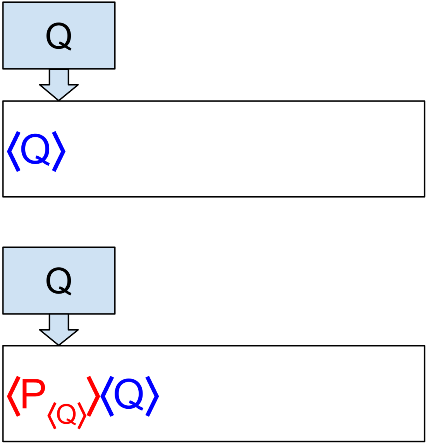

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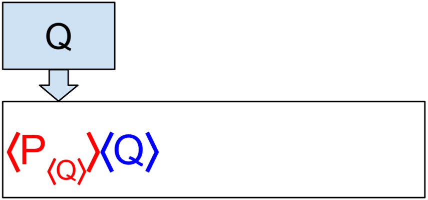
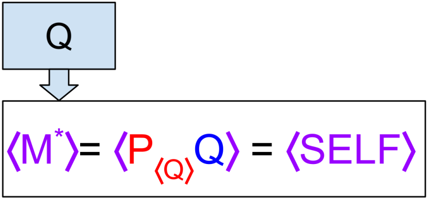
4. **Q combines $\langle P_{\langle Q \rangle} \rangle$ and $\langle Q \rangle$ into a single machine $\langle M^{*} \rangle$**





Self-analyzing programs

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$$r(w) = t(\langle R \rangle, w)$$

Self-analyzing programs

- Let T be a Turing machine that computes a function $t : \Sigma^* \times \Sigma^* \to \Sigma^*$.
  - T represents a machine that receives a machine description as one of its inputs and analyzes that machine
- There is a Turing machine R that computes a function $r : \Sigma^* \to \Sigma^*$, where for every w
$$r(w) = t(\langle R \rangle, w)$$
  - R performs the same analysis as T does, but it analyzes its own description rather than taking a machine description as one of its inputs

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2. Q reads $\langle QT \rangle$ on the tape and uses this to construct $P_{\langle QT \rangle}$.
3. Q then combines $\langle P_{\langle QT \rangle} \rangle$, $\langle Q \rangle$, and $\langle T \rangle$ into one machine $\langle M^* \rangle = \langle P_{\langle QT \rangle} QT \rangle = \langle R \rangle$

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4. Q writes $\langle R \rangle$ onto the tape (again, following w) and passes control to T
5. T reads w and $\langle R \rangle$ on. the tape and computes $t(\langle R \rangle, w)$

Self-analyzing programs

$R = \langle P_{\langle QT \rangle} QT \rangle$

Initially $P_{\langle QT \rangle}$ starts with control of the tape, with an input string w

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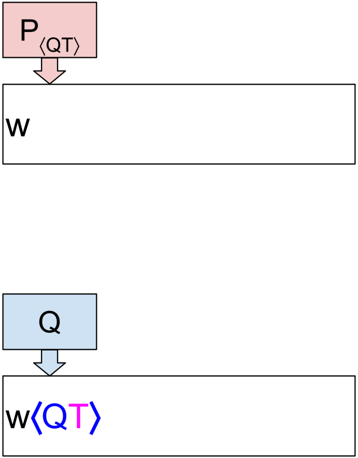

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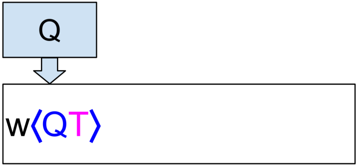

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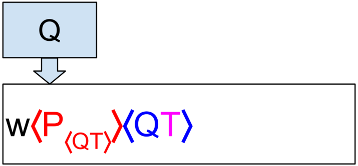

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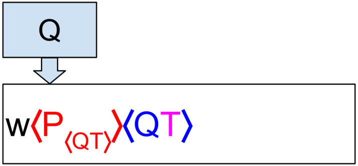

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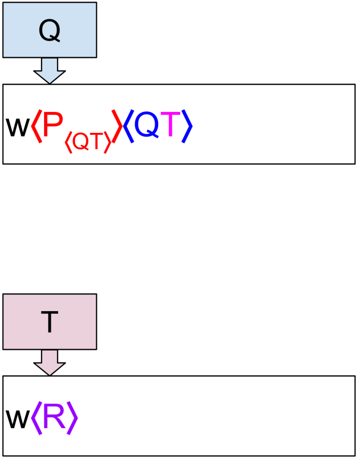

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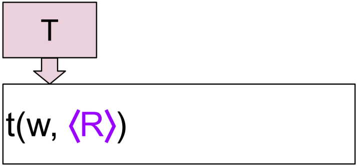

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- When constructing a Turing machine M, you can include the command "Obtain M's description $\langle M \rangle$" as part of the pseudocode
- Whatever you were going to do with $\langle M \rangle$, the recursion theorem tells us that there exists a machine that finds a way to put its own description on the tape before processing that description

HALT is undecidable: another proof

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Theorem: $\mathrm{HALT} = \{\langle M, w\rangle | M \text{ halts on } w\}$ is undecidable

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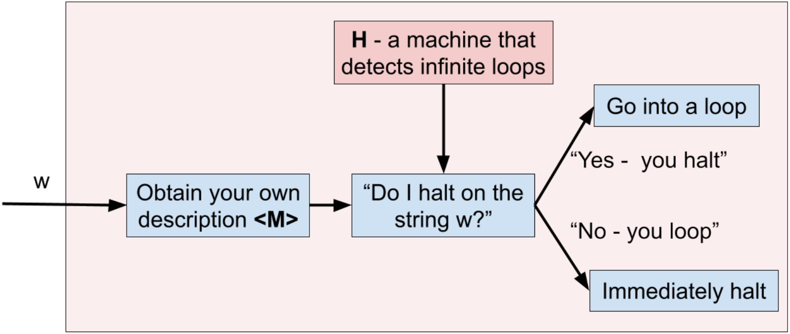
M halts on input w if H says it should loop, and loops if H says it should halt.

HALT is undecidable: another proof

HALT = {<M, w> | M halts on w}
A paradoxical machine **M**



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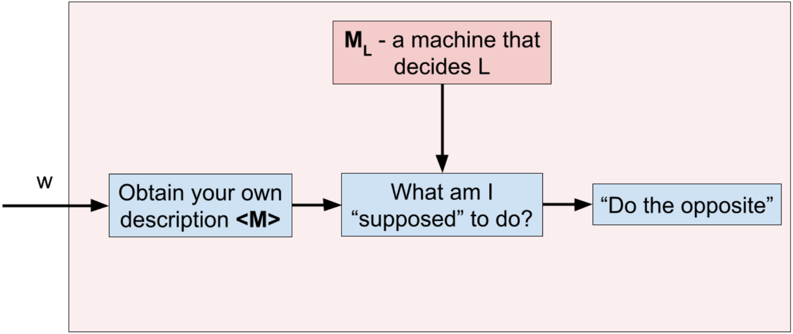
We can use the following recipe to prove that a language L is undecidable

1. AFSOC a machine D can decide L
2. Create a machine M that obtains its own description, uses D to analyze itself, and "does the opposite" of what it should
3. Conclude that D is not deciding L correctly

Recursion theorem recipe

L = an undecidable language

A paradoxical machine **M**



If we can decide L, we create a paradoxical machine

A_{TM} is undecidable: an alternate proof

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Theorem: $\mathrm{A_{TM}} = \{\langle M, w\rangle | w \in L(M)\}$ is undecidable

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Let's try proving this using the recursion theorem

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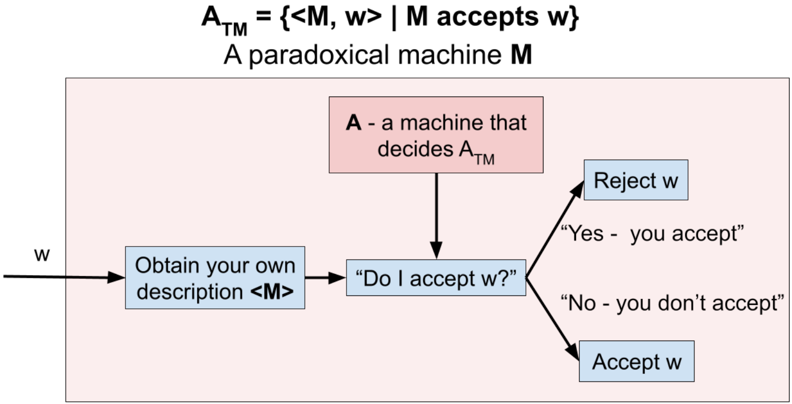

If we can decide A_{TM}, we create a paradoxical machine

The language $\mathrm{REG}_{\mathrm{TM}}$

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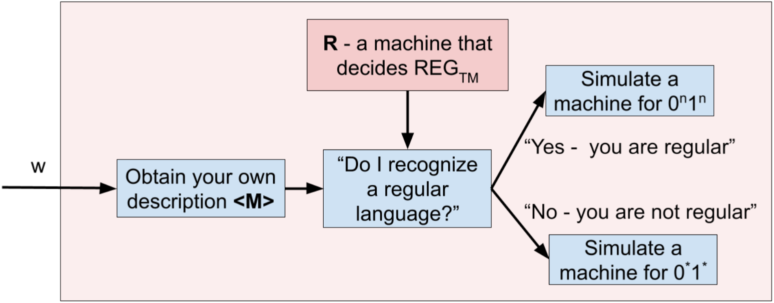
- If M says A is not regular, it recognizes 0^*1^* - a regular language.
- Thus, R is not deciding $\mathrm{REG_{TM}}$ correctly

$\mathrm{REG}_{\mathrm{TM}}$ is undecidable

$\mathrm{REG}_{\mathrm{TM}}$ = {<M> | L(M) is regular}

A paradoxical machine **M**



If we can decide $\mathrm{REG}_{\mathrm{TM}}$, we create a paradoxical machine

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i.e., there is no machine with a shorter description than M that recognizes the same language.

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  - "Can this source code be rewritten more concisely?"

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4. Simulate M_2 on w

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- But this is not supposed to be possible since M_2 is supposed to be a minimal TM.
- Thus, our enumerator did not enumerate $\mathrm{MIN}_{\mathrm{TM}}$ correctly.