Regular Expressions

Arjun Chandrasekhar

Regular Expressions

- Another way to describe languages
- A formula that can be used to generate strings
- Formed by combining smaller regular expressions using the three regular operations

Regular Expressions

Let Σ be a alphabet. We say R is a **regular expression** if R is:

1. σ for some $\sigma \in \Sigma$
2. ϵ
3. $\emptyset$
4. $R_1 \cup R_2$, where R_1 and R_2 are regular expressions
5. $R_1 \circ R_2$, where R_1 and R_2 are regular expressions, or
6. R_1^* where R_1 is a regular expression

Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

$$R = a$$

A) ϵ (empty string)

B) a

C) b

D) aaaaaaaaaa

E) None of the above

F) invalid regex

Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

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A) ϵ (empty string)

B) a ✓

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Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

$$R = \epsilon$$

A) ϵ (empty string)

B) a

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E) None of the above

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Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

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Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

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Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

$$R = \emptyset$$

A) ϵ (empty string)

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Empty set is different from empty string!

Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

$$R = a \circ b^*$$

A) ϵ (empty string)

B) a

C) abababab

D) abbbbbbb

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Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

$$R = (a \circ b)^*$$

A) ϵ (empty string)

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Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

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B) ab ✓

C) abababab ✓

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Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

$$R = (a \circ b)^* \circ b \circ b \circ a$$

A) ϵ (empty string)

B) bba

C) abab

D) ababbba

E) None of the above

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Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

$$R = (a \circ b)^* \circ b \circ b \circ a$$

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Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

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Regular expressions

Let $\Sigma = \{a, b\}$ Which strings are generated by the following regex?

$$R = (a \circ b^*) \cup (c \circ d^*)$$

A) ϵ (empty string)

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$$R = (a \circ b^*) \cup (c \circ d^*)$$

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D) b

E) None of the above

F) invalid regex ✓

Regular Expressions

Some notes:

- Don't confuse Kleene star and linux wildcard
- Don't confuse ϵ and $\emptyset$; they are different
- $L(R)$ is the *language of R*, i.e. the set of strings generated by R
- The operator precedence is $* > \circ > \cup$
  - Parentheses can be used to override this
- Concatenation is often done implicitly
  - Can write bba instead of $b \circ b \circ a$
- Σ is often shorthand for $\sigma_1 \cup \sigma_2 \ldots \sigma_n$
- R^+ is shorthand for RR^* (which is shorthand for $R \circ R^*$)

Applications of Regular Expressions

- Lexical-analyzer generators, such as lex and flex. A lexical-analyzer is the part of a compiler that breaks a program into tokens. Regular expressions specify the valid tokens of a programming language.
- String search tools that are built into operating system utilities (like awk and grep in Unix), text editors, and programming language libraries. Regular expressions describe the strings that are being searched for.
- The regular expressions in these tools typically have a richer set of operators, to facilitate more easily describing strings.

Example regexes

Let $\Sigma = \{0, 1\}$

1. $\Sigma^* = (0 \cup 1)^*$ = all binary strings (you have already seen this one!)
2. $\Sigma^* 001 \Sigma^* = \{$w $|$ w contains 001 as a substring$\}$
3. $0 \cup 1 \cup (0\Sigma^*0) \cup (1\Sigma^*1) = \{$w $|$ w starts and ends with the same symbol$\}$

Regex practice

Let $\Sigma = \{0, 1\}$. Write a regex for:

$$\mathsf{L} = \{\mathsf{w} \mid \text{every odd position is } 1\}$$

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$$R = (1\Sigma)^* \circ (\epsilon \cup 1)$$

Regex practice

Let $\Sigma = \{0, 1\}$. Write a regex for:

$$\mathsf{L} = \{\mathsf{w} \mid \mathsf{w\ contains\ an\ even\ number\ of\ 0s}\}$$

Regex practice

Let $\Sigma = \{0, 1\}$. Write a regex for:

$$L = \{w \mid w \text{ contains an even number of 0s}\}$$

$$(1^*01^*0)^*1^*$$

$$1^*(01^*01^*)^*$$

Regex practice

Let $\Sigma = \{0, 1\}$. Write a regex for:

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$$(0^*1)(0^*1)0^*$$

Regex practice

Let $\Sigma = \{0, 1\}$. Write a regex for:

$$L = \{w \mid \text{w contains an even number of 0s or exactly two 1s}\}$$

Regex practice

Let $\Sigma = \{0, 1\}$. Write a regex for:

L $= \{$w $|$ w contains an even number of 0s or exactly two 1s$\}$

$$((1^*01^*0)^*1^*) \cup (0^*10^*10^*)$$

Regex practice

Let $\Sigma = \{0, 1\}$. Write a regex for:

L $= \{$w $|$ w contains an even number of 0s and exactly two 1s$\}$

Regex practice

Let $\Sigma = \{0, 1\}$. Write a regex for:

$$L = \{w \mid \text{w contains an even number of 0s and exactly two 1s}\}$$

$$(00)^*1(00)^*1(00)^*$$
$$\cup$$
$$0(00)^*10(00)^*1(00)^*$$
$$\cup$$
$$(00)^*10(00)^*10(00)^*$$
$$\cup$$
$$0(00)^*1(00)^*10(00)^*$$

Regex to NFA

Design an NFA with 5 states to recognize $0^* \cup 1^*0^+$

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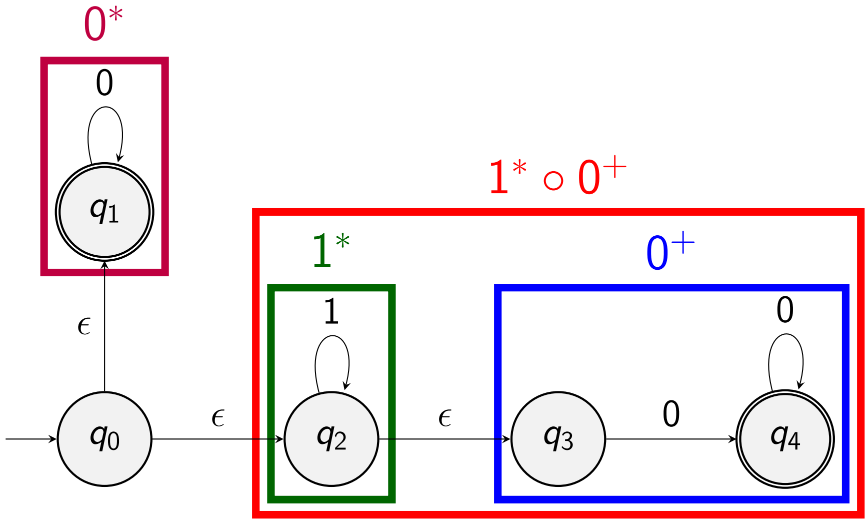


Kleene's Theorem

Theorem: A language is described by a regular expression if and only if it is regular

- What are the two directions we must prove?
  - ($\Rightarrow$) If a language is described by a regular expression, it is regular
  - ($\Leftarrow$) If a language is regular, then it is described by a regular expression
- **Recall:** A language is regular if and only if it is described by a DFA
  - Or equivalently (and conveniently), an NFA

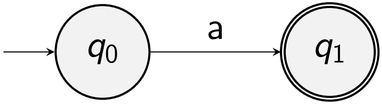
Kleene's Theorem (Forward Direction)

Claim: If a language L can be described by a regular expression R, then L is regular

- **Proof Idea:** We will use induction to create an NFA for R
- Show how to make an NFA for the atomic regular expressions
- For union, concatenation, and star, use induction to make NFAs for the smaller parts of the expression, and then combine them

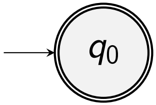
Kleene's theorem (Forward Direction)

Base Case: $R = \sigma \in \Sigma$



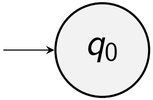
Kleene's theorem (Forward Direction)

Base Case: $R = \epsilon$



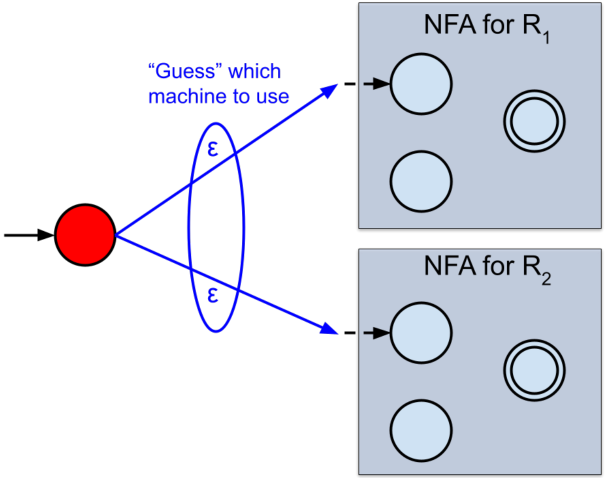
Kleene's theorem (Forward Direction)

Base Case: $R = \emptyset$



Kleene's theorem (Forward Direction)

Inductive Case: $R = R_1 \cup R_2$



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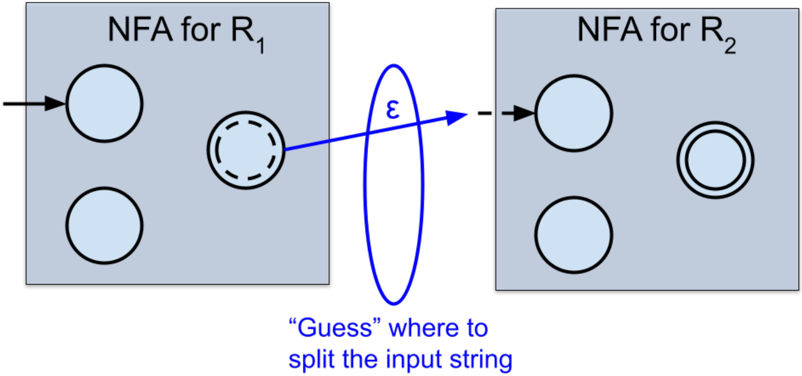
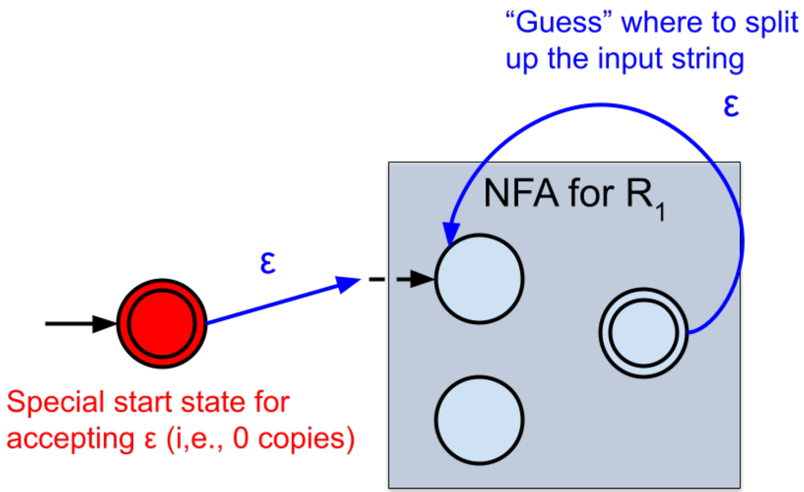


Kleene's theorem (Forward Direction)

Inductive Case: $R = (R_1)^*$



Regex to NFA Conversion Example

Let's make an NFA for $R = ((ab) \cup a)^*$

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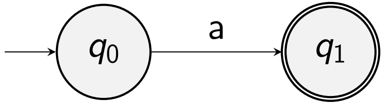
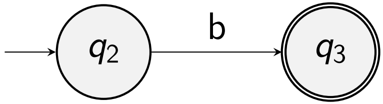
NFA for a



Regex to NFA Conversion Example

Let's make an NFA for $R = ((ab) \cup a)^*$

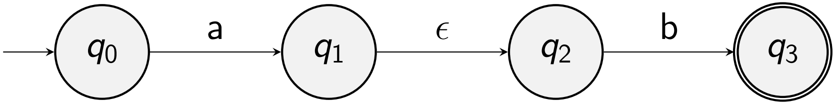
NFA for b



Regex to NFA Conversion Example

Let's make an NFA for $R = ((ab) \cup a)^*$

NFA for $ab = a \circ b$


$$\begin{tikzcd} \arrow[r] & q_0 \arrow[r, "a"] & q_1 \arrow[r, "\epsilon"] & q_2 \arrow[r, "b"] & \boxed{q_3} \end{tikzcd}$$

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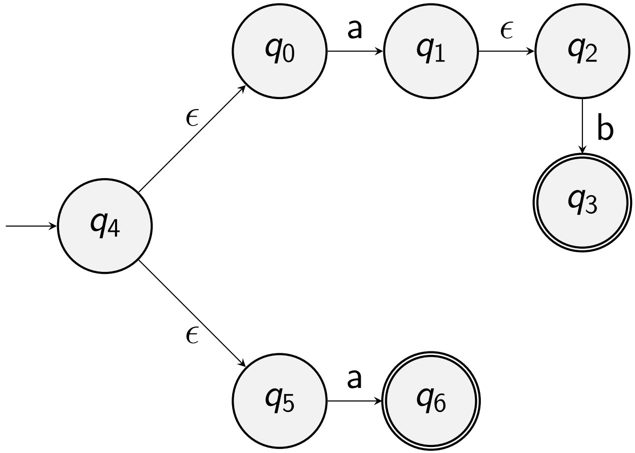
NFA for $ab \cup a$



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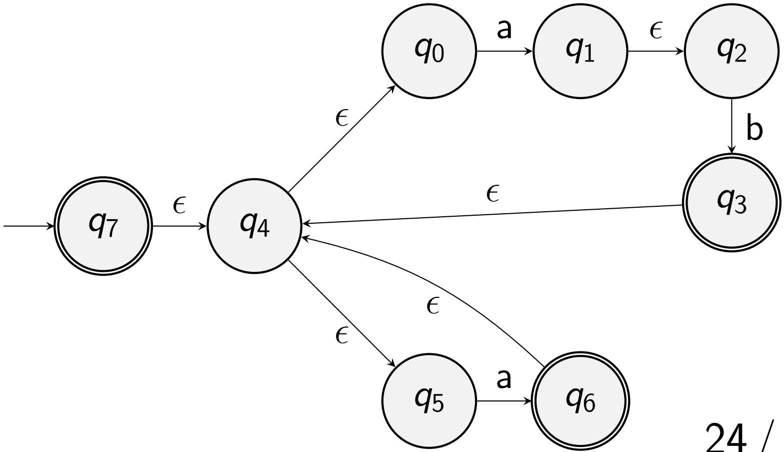

$$\begin{tikzcd}
 & & q_0 \arrow[r, "a"] & q_1 \arrow[r, "\epsilon"] & q_2 \arrow[d, "b"] \\
\arrow[r] & q_7 \arrow[r, "\epsilon"] & q_4 \arrow[u, "\epsilon"] \arrow[d, "\epsilon"] & & q_3 \arrow[ll, "\epsilon"'] \\
 & & q_5 \arrow[r, "a"] & q_6 \arrow[ul, bend right, "\epsilon"'] &
\end{tikzcd}$$

Kleene's Theorem (backwards direction)

Claim: If L is regular, then L can be described by a regular expression

- **Proof Idea**: Convert the DFA for L into a regex
- Extend the DFA so that each transition is a regex
- "Rip" states one at a time, and modify the other transitions to compensate
- When there's just one transition remaining, we will have the desired regex

GNFAs

A **Generalized Nondeterministic Finite Automata (GNFA)** is an NFA in which arrows are labelled by regular expressions (rather than symbols)

$$\begin{tikzcd}
& q_1 \arrow[loop above, "ab"] \arrow[rr, "(aa)^*"] \arrow[rrd, "b^*"] & & q_2 \arrow[loop above, "aa"] \arrow[ll, bend right, "a^*"'] \arrow[d, "ab \cup ba"] \\
{} \arrow[r] & q_0 \arrow[u, "\emptyset"] \arrow[rru, "ab^*"] \arrow[rr, "b"] & & q_3
\end{tikzcd}$$

GNFAs

For convenience we require GNFAs be in the following special form:

- The start state q_s has transition arrows going to every other state, but no arrows coming in from any other state
- There is only a single accept state, q_F, and it has arrows coming in from every other state but no arrows going to any other state. Furthermore, the accept state is not the same as the start state.
- Except for the start and accept states, one arrow goes from every state to every other state, and also from each state to itself.

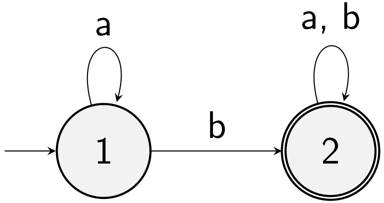
DFA to GNFA

To make a DFA onto a GNFA:

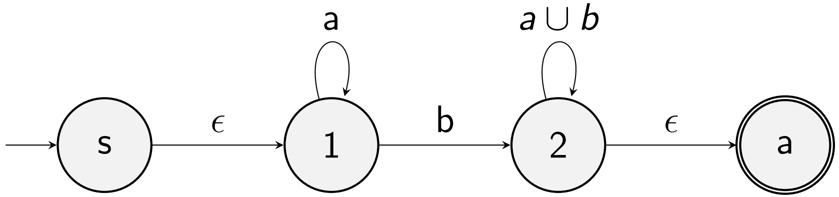
1. Create a special start state, with an ϵ transition to the original start state
2. Add a special accept state, with ϵ transitions from the original accept states
3. If any transition has multiple symbols, combine them into a union regex
4. If any transition between states is missing, ad an $\emptyset$ transition
    - We can omit these when drawing state diagrams

DFA to GNFA

Starting DFA

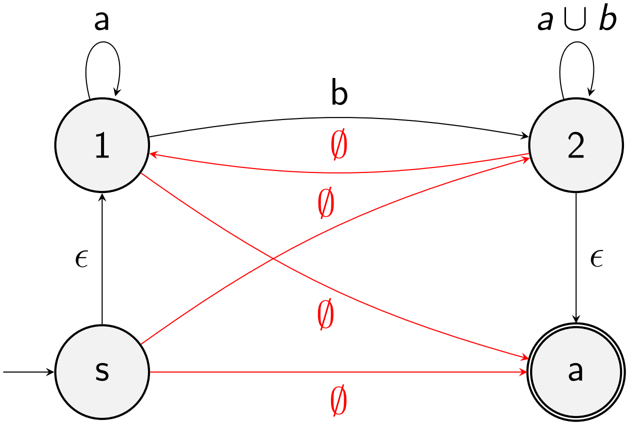


Starting GNFA



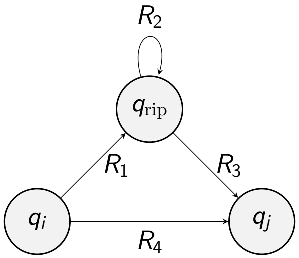
DFA to GNFA

Full starting GNFA



Ripping a state

How could we get from q_i to q_j?



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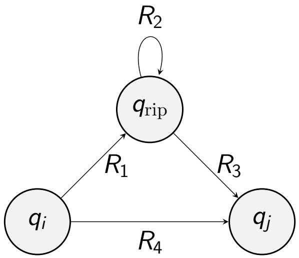
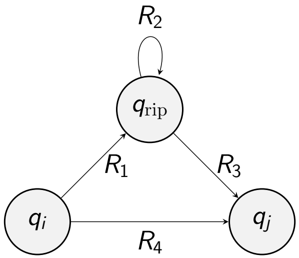


1. Go directly from q_i to q_j
2. Go through q_{rip}
   2.1 $q_i \to q_{\text{rip}}$
   2.2 $q_{\text{rip}} \to q_{\text{rip}}$ any number of times
   2.3 $q_{\text{rip}} \to q_j$

Ripping a state

How could we get from q_i to q_j?



1. Go directly from q_i to q_j (R_4)
2. Go through q_{rip} $(R_1 \circ R_2^* \circ R_3)$
   - 2.1 $q_i \rightarrow q_{\mathrm{rip}}$ (R_1)
   - 2.2 $q_{\mathrm{rip}} \rightarrow q_{\mathrm{rip}}$ any number of times (R_2^*)
   - 2.3 $q_{\mathrm{rip}} \rightarrow q_j$ (R_3)

$R = (R_1 \circ R_2^* \circ R_3) \cup R_4$

Ripping a State

Before

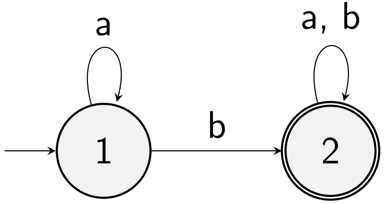
$$\begin{tikzcd}
 & q_{\text{rip}} \arrow[loop above, "R_2"] \arrow[dr, "R_3"] & \\
q_i \arrow[ur, "R_1"] \arrow[rr, "R_4"'] & & q_j
\end{tikzcd}$$

After

$$\begin{tikzcd}
q_i \arrow[r, "(R_1 \circ R_2^* \circ R_3) \cup R_4"] & q_j
\end{tikzcd}$$

DFA to Regex

Starting DFA



DFA to Regex

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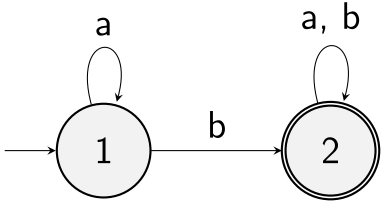
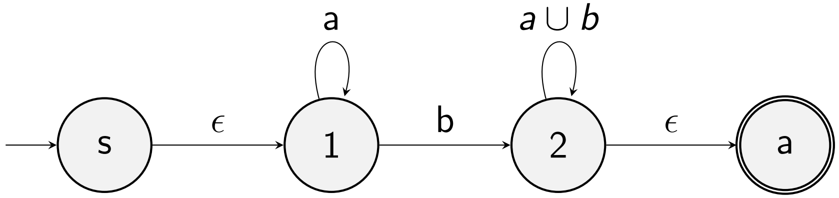
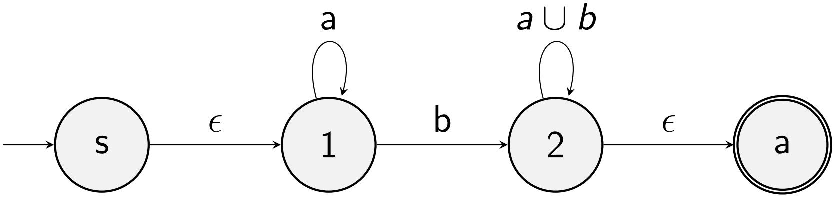


Starting GNFA

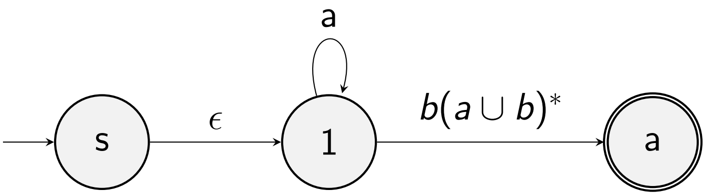


DFA to Regex

Starting GNFA

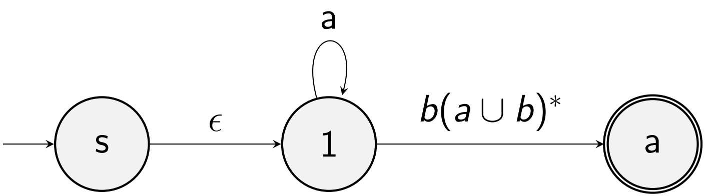


Rip State 2

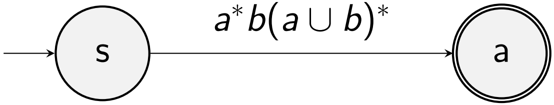


DFA to Regex

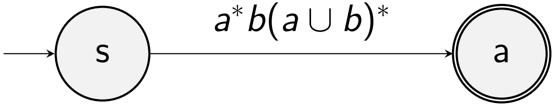
Rip State 2



Rip State 1



DFA to Regex



$$R = a^*b(a \cup B)^*$$

Regular expressions recap

- Regular expressions are equivalent to NFAs
    - Which makes them equivalent to DFAs
- DFAs are equivalent to regular expressions

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▶

▶

Regular expressions recap

- Regular expressions are equivalent to NFAs
  - Which makes them equivalent to DFAs
- DFAs are equivalent to regular expressions
- A language is regular if and only if it is described by a regular expression
- To show a language is regular, can use a state machine or a regex

Regular expression closure proofs

- Regular expressions characterize the regular languages
- To show a language is regular, it is sometimes more convenient to use a regex than a state machine
- Sometimes, it is easier to write a closure proof using a regex
- **Blueprint:** Use an inductive proof
  - **Base case:** Show that closure holds for the three atomic regexes ($\emptyset, \epsilon, a$)
  - **Inductive case:** show that closure holds for t he three regular operations (union, concatenation, Kleene star)

EVERY-OTHER closure

Claim: If A is regular then $\text{EVERY-OTHER}(A)$ is regular

$$\text{EVERY-OTHER}(A) = \{w = \begin{array}{l} a_1 y_1 \ldots a_n y_n | \\ a_1 \ldots a_n \in A \\ y_i' s \text{ can be anything}\} \end{array}$$

- Because A is regular, it is described by a regular expression R
- We will construct a regular expression R' that describes $\text{EVERY-OTHER}(A)$

EVERY-OTHER closure: base case

- $R = \emptyset$
  $R' = \emptyset$
- $R = \epsilon$
  $R' = \epsilon$
- $R = a \in \Sigma$
  $R' = a\Sigma$

EVERY-OTHER closure: inductive case

Assume if A is described by a regex R_i with size $\leq n$, there is a regex R_i' for $\text{EVERY-OTHER}(A)$

Let A be described by a regex R with size $n+1$.

- $R = R_1 \cup R_2$
  $R' = R_1' \cup R_2'$
- $R = R_1 \circ R_2$
  $R' = R_1' \circ R_2'$
- $R = (R_1)^*$
  $R' = (R_1')^*$