Theory of Computation//The Halting Problem

Arjun Chandrasekhar

Unsolvable Problems

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strange.java checks if its own source code has an even length

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- No computer program will EVER solve these problems
- We will make use of diagonalization, as well as machines that take other machines as input
  - "If we could recognize this language, we could construct a that machine contradicts every machine in the world - including itself"

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- **Proof idea:** if we could do this, we could write a program that literally contradicts itself
- We will write a program that runs this compiler on itself and then does the opposite of what it is "supposed" to do
- Our program will "fool" the compiler, thus proving the compiler doesn't actually perform as advertised

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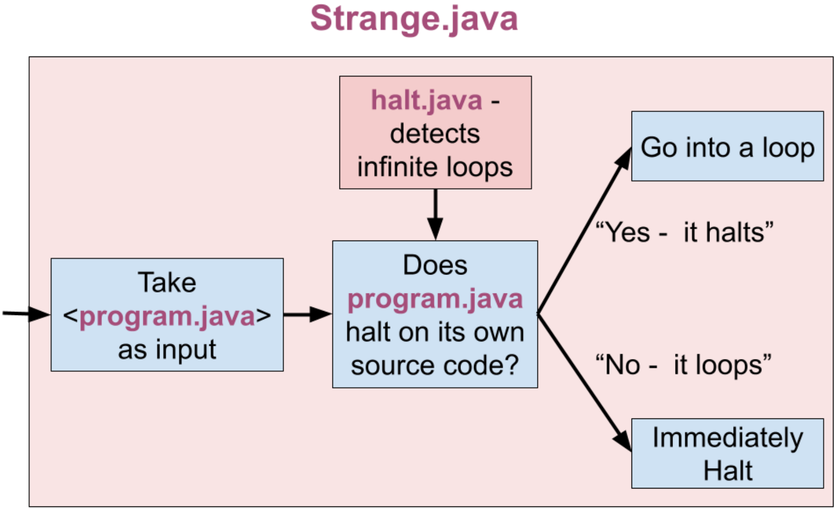

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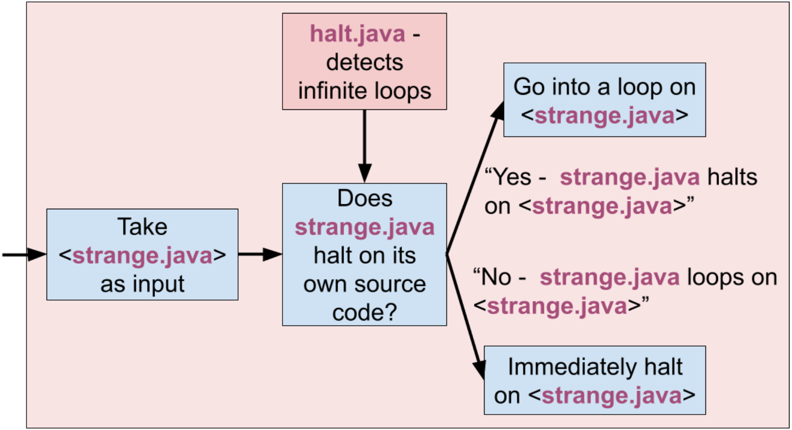
Let's feed **Strange.java** its own source code



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- We conclude that halt.java is not detecting infinite loops correctly.

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- It's only possible to create strange.java if we assume that halt.java exists
- We conclude that halt.java doesn't exist, because paradoxical programs don't exist

HALT is Undecidable

Let's prove the same theorem using Turing machines

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- By assuming that H exists, we can create a machine S that should not exist

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- Think of $\langle M \rangle$ as a source code file and M as an actual executable that can be run

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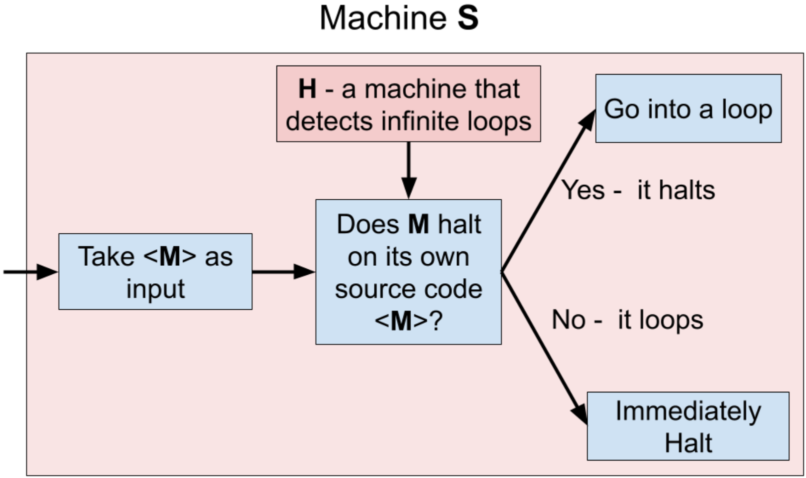

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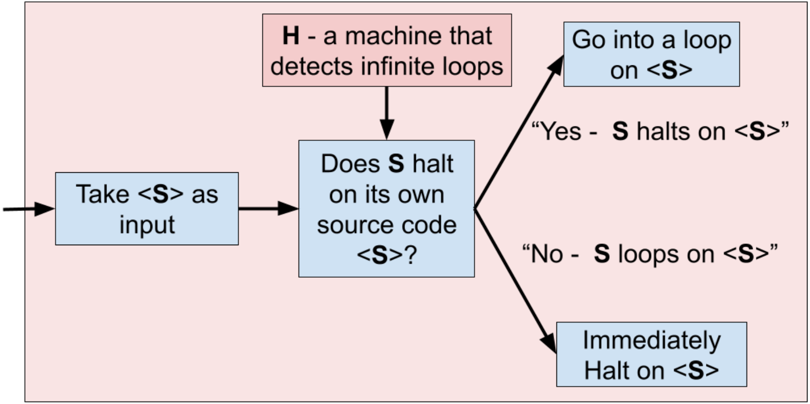

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There is no way that H is actually deciding HALT correctly!

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- We assumed that we could determine what every program does on every possible input
- We constructed a machine S that contradicted every program in the universe
  - But this means that S contradicts itself
- Thus we reject our original assumption

Diagonalizing HALT

	$<M_1>$	$<M_2>$	$<M_3>$	...	$<S>$
M_1	**HALT**	LOOP	LOOP	...	HALT
M_2	LOOP	**LOOP**	LOOP	...	HALT
M_3	HALT	LOOP	**HALT**	...	LOOP
...	...	...	...	...	...
S	LOOP	HALT	LOOP	...	???

Box (i, j):
"Does machine M_i halt or loop on source code $<M_j>$"?

The existence of **H**, a decider for **HALT**, allows us to fill in this table (and then construct the paradoxical machine S)

Construct **S** by taking the "opposite" of the **diagonals** until we reach a **contradiction**

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- If M halts on w then H needs to accept $\langle M, w \rangle$
- If M loops on w then H should reject
  or possibly loop on $\langle M, w \rangle$

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- If M does indeed halt on w then eventually H will accept $\langle M, w \rangle$
- If M loops forever on w, H will do the same, so it will not accept $\langle M, w \rangle$ (which is sufficient)

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co-Recognizable Languages

A language L is **co-Turing Recognizable** if it's complement $\overline{L}$ is recognizable

- We can construct a machine $\overline{M}$ that recognizes $\overline{L}$
- If $w \in \overline{L}$ (i.e. $w \notin L$) then $\overline{M}$ will halt and accept
- If $w \notin \overline{L}$ (i.e. $w \in L$) then $\overline{M}$ will reject or loop forever

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- **Note:** In prior lectures we used L^c to denote the complement. For these topics, the convention is to use $\overline{L}$ to denote the complement

co-Recognizable Languages

Theorem: A language is decidable if and only if L is both recognizable and co-recognizable

1. ($\Rightarrow$) If a language is decidable it is both recognizable and co-recognizable
2. ($\Leftarrow$) If a language is both recognizable and co-recognizable, it is decidable

co-Recognizable Languages

($\Rightarrow$) If L is decidable then it is recognizable

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- Let M be the machine that decides L

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- Let M be the machine that decides L
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co-Recognizable Languages

($\Rightarrow$) If L is decidable then it is recognizable

- Let M be the machine that decides L
- Then M also recognizes L!
  - M always halts
  - If $w \in L$ then M will halt and accept
  - If $w \notin L$, M will not accept (in fact, it will halt and reject)

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co-Recognizable Languages

($\Rightarrow$) If L is decidable then it is co-recognizable

- Let M be the machine that decides L
- To recognize $\overline{L}$ we create a machine $\overline{M}$ that runs M and does the opposite
  - M always halts, so $\overline{M}$ always halts
  - If $w \in \overline{L}$ then M will halt and reject, so $\overline{M}$ will halt and accept
  - If $w \notin \overline{L}$, then $w \in L$. So M will halt and accept, and $\overline{M}$ will halt and reject

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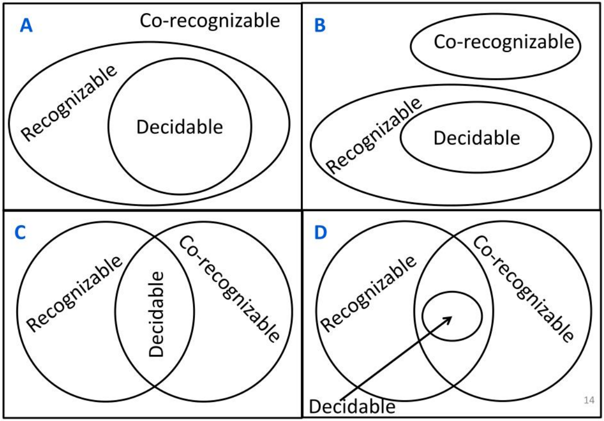
  3. If $\overline{M}$ accepts, reject
- Exactly one of the two machines has to eventually accept, so D always halts

Decidable vs. (co-)Recognizable Languages

So, what does the Venn Diagram look like?



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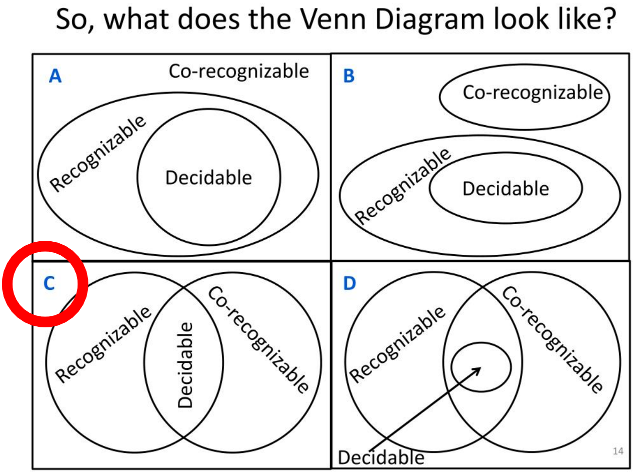

The language $\overline{\mathrm{HALT}}$

Consider the following language

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- Pronounced "co-HALT"
- If M loops on w we accept $\langle M, w\rangle$
- If M halts on w we reject $\langle M, w\rangle$
- $\overline{\mathrm{HALT}}$ is co-recognizable because its complement HALT is recognizable

Unrecognizability of $\overline{\mathrm{HALT}}$

Theorem: $\overline{\mathrm{HALT}}$ is not Turing-recognizable

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- Then HALT is co-recognizable

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- We know that HALT is also recognizable

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- Then HALT is co-recognizable
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- Then HALT would decidable, which is a contradiction!
- We conclude that $\overline{\mathrm{HALT}}$ is unrecognizable